On Gaudí's Magic Square

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Abstract

The authors present from a heuristic viewpoint an elementary study of the unusual magic square (GMS) found in the outstanding Gaudí's Sagrada Famìlia Temple in Barcelona. Its magic sum is 33 and it features on rows, columnns, diagonals and 2/2 broken diagonals, and on many 2×2 subsquares, such as the central one or the one formed by the entries at the four corners. Elementary Group Theory is introduced as a natural tool in the study of the magic square in order to analyse how these magic-sum preserving structures wander on the torus $(\mathbf{Z}/4) \times (\mathbf{Z}/4)$, the geometric counterpart of the plane square.

1 Introduction

Magic squares are found quite often in recreational Mathematics and are used as teaching aides in various arithmetical questions. An n-th order magic square is an $n \times n$ array of whole numbers having the property that the sum of the terms in any row, in any column and in any of the two diagonals is the same. This common sum M is usually called the magic sum of the square. It is customary that the numbers in the square be part of some arithmetic progression, and the case $\{1, 2, 3, ..., n^2\}$ is the standard one for the $n \times n$ square. Nevertheless, this is not compulsory and repeated numbers are allowed, as well as apparently random chosen elements. Magic-square practitioners are able to build squares with many different kinds of predetermined magic-sum preserving subsets, *e. g.* having the shape of letters or other figures, but here we shall concentrate on simpler matters. See the classic well-known book by Andrews [1], the excellent webpage www.magic-squares.de, and many articles in the electronic journal of mathematical enrichment at nrich.maths.org.

During a recent trip to Barcelona on the occasion of the sesquicentennial year dedicated to the famous architect Antoni Gaudí who lived between 1852 and 1926 (see [2] for a quick and colourful reference), the authors observed in one of the façades of his still unfinished Sagrada Famìlia Temple (Figure 1) the magic square shown in Figure 2. It is a large artwork over $1m \times 1m$, located in the wall, at the left of the doorway. It is rather curious that an Internet search for "magic square gaudi" yielded only four results with little mathematical interest. A rather complete and most interesting account on GMS can be read in Maritz's paper [3]. The magic sum of this fourth order square is $M = 33$ row-, column-, and diagonalwise: It is a real magic square. In a first glimpse, it is somehow displeasing that neither 12 nor 16 appear in it, while 14 and 10 are shown twice. But this first impression quickly vanishes when a wealth of magic-sum preserving structures is discovered.

2 Magic Squares and Symmetries on the Torus

Within a magic square, rows, columns and the two main diagonals are spatial structures with n elements each (a single cell of the square would be a one-element structure) upon which the magic sum is preserved. For our purpose, it is better to consider them as the result of some moving n-windows that select n items out of the available n^2 numbers. Let $S(W)$ the sum of the n numbers selected by a general n-window. Then the definition of a magic square amounts to the following:

Definition 1 *An $n \times n$ array of whole numbers is called a magic square when $S(W) = M$ holds for any n-window $W \in \{Rows, Columns, Diagonals\}$.*

Indeed there can exist n-windows other than the usual rows, columns and diagonals with the property $S(W) = M$. Magic squares having these magic-sum preserving spatial structures are given different names, all of them in the "more-than-magic square" mood. An investigation of these features can be better undertaken by trying some geometry on magic squares.

Let R be the row-like n-window. If R is displaced up and down, the sum of the selected row is indeed M, for otherwise the square would not be a magic one. To add some mathematical flavour, consider only the downward displacement and note that there exist exactly n consecutive one-step displacements of R before coming back to the row shown by R in the first instance. In other words, the following obvious proposition on a symmetry (in the general sense) of magic squares is obtained:

Proposition 2 *Let $g(R)$ denote the translate of R by any $g \in \mathbf{Z}/n$ under the group action just defined. Then $S(g(R)) = M$.*

Indeed, the above proposition is also true for the column-like window C. Thus, it is shown that S is an invariant quantity under the action of some displacement groups operating on certain classes of n-windows. This is pure *à la* Klein geometry, so let us build a natural object where group actions will be readily interpreted.

Identification –glueing– of opposite sides of the square yields a torus, where things become more conveniently displayed. For instance, the 4 entries at the four corners of the original square arrange themselves neatly in a 2×2 square on the torus surface, and the main diagonals, as well as the broken diagonals (which become no longer broken ones), wrap helically around the torus.

Now let us consider the group $G_n = (\mathbf{Z}/n) \times (\mathbf{Z}/n)$ acting as a translation group on the square in the following way: $(g, h) \in G_n$ acts on a single cell by translating it say g units to the right and h units upwards. The torus identification allows the wandering 1-window to visit once and again all locations on the square without leaving the torus surface. The action of (g, h) on a general n-window is achieved by the joint action upon the individual cells of the window. The next Proposition on magic-sum preserving toroidal symmetries follows immediately:

Proposition 3 *Let $(g, h)\,(W)$ denote the translate by $(g, h) \in G_n$ of any row- or column-like n-window W under the group action just defined. Then $S((g, h)\,(W)) = M$.*

This result shows that the row-wise (as well as the column-wise) magic sum is a group invariant of G_n and any of its subgroups. The idea of a subgroup action becomes helpful in the treatment of the diagonal sums:

Proposition 4 *Let us consider the diagonal subgroup $\{(g, g) \in G_n \mid g \in \mathbf{Z}/n\} \simeq \mathbf{Z}/n$. If D is a main diagonal n-window, then $S((g, g)\,(D)) = M$.*

This proposition simply states that the magic sum of a principal diagonal is preserved under any translation along its direction.

3 Mathematically writing on GMS

Now, let us turn our attention to the magic square in Gaudí's Cathedral. Figure 2 shows the magic square, which is copied here for quick reference:

1	14	14	4
11	7	6	9
8	10	10	5
13	2	3	15

In this case, $n = 4$ and $G_4 = \mathbf{Z}/4 \times \mathbf{Z}/4$. The group operation on G_4 is defined as componentwise addition mod 4. The possible nontrivial subgroups of G_4 have orders 2, 4, and 8, and the complete list is:

- Three order 2 subgroups:
  $H_{2,1} = \{(0,0), (0,2)\} \simeq \mathbf{Z}/2$
  $H_{2,2} = \{(0,0), (2,0)\} \simeq \mathbf{Z}/2$
  $H_{2,3} = \{(0,0), (2,2)\} \simeq \mathbf{Z}/2$
- Four order 4 subgroups:
  $H_{4,1} = \{(0,0), (0,2), (2,0), (2,2)\} \simeq V_4$ (Klein's group).
  $H_{4,2} = \{(0,0), (0,1), (0,2), (0,3)\} \simeq \mathbf{Z}/4$
  $H_{4,3} = \{(0,0), (1,0), (2,0), (3,0)\} \simeq \mathbf{Z}/4$
  $H_{4,4} = \{(0,0), (1,1), (2,2), (3,3)\} \simeq \mathbf{Z}/4$

- Three order 8 subgroups:

  $H_{8,1} = \{(0,0),(0,2),(1,1),(1,3),(2,0),(2,2),(3,1),(3,3)\}$
  $H_{8,2} = \{(0,0),(0,2),(1,0),(1,2),(2,0),(2,2),(3,0),(3,2)\}$
  $H_{8,3} = \{(0,0),(0,1),(0,2),(0,3),(2,0),(2,1),(2,2),(2,3)\}$

3.1 Square 4-windows

We are interested in square shaped 4-windows, *i.e.* 2×2 subsquares. Direct inspection shows that *e.g.* the four elements in the bottom left 2×2 square

8	10
13	2

also have the magic sum 33. Now, let the action of G_4 displace this square 4-window on the torus surface. A simple check shows that not every element of G_4 will preserve the magic sum: Just observe that under $(1,0)$

<table><tr><td><table><tr><td>8</td><td>10</td></tr><tr><td>13</td><td>2</td></tr></table></td><td>action of $(1,0)$ $\longrightarrow$</td><td><table><tr><td>10</td><td>10</td></tr><tr><td>2</td><td>3</td></tr></table></td></tr></table>

the magic sum 33 is lost. The same can be said about the action of $(0,1)$, $(3,0)$, $(0,3)$,... Therefore, the magic sum is not preserved under the action of subgroups including any of these elements. However, the actions of $(2,0)$, $(0,2)$, $(1,1)$, $(2,2)$, $(3,3)$,... do preserve the magic sum. A little book-keeping work (Figure 3) could be summed up in the following piece of mathematical writing:

Definition 5 *Let $(g,h) \in G_4$. The index of (g,h) is defined as the* $\bmod 2$ *residual class of $g+h$.*

Proposition 6 *The magic sum $M = 33$ of square structures in GMS is preserved under the action of the elements of G_4 having zero index.*

This last proposition can be restated in terms of subgroups of G_4:

Proposition 7 *The magic sum $M = 33$ of a square structure in GMS is preserved under the action of the proper subgroups $H_{2,1}$, $H_{2,2}$, $H_{2,3}$, $H_{4,1}$, $H_{4,4}$, and $H_{8,1}$.*

The action of elements with index 1 is most interesting as well. For instance, let us remind that $3 = -1 \bmod 4$ and consider the following two inverse actions on the bottom left square:

<table><tr><td><table><tr><td>5</td><td>8</td></tr><tr><td>15</td><td>13</td></tr></table></td><td>$(3,0)$ $\longleftarrow$</td><td><table><tr><td>8</td><td>10</td></tr><tr><td>13</td><td>2</td></tr></table></td><td>$(1,0)$ $\longrightarrow$</td><td><table><tr><td>10</td><td>10</td></tr><tr><td>2</td><td>3</td></tr></table></td></tr></table>

The 33 sum of our old friend is changed into 25 under $(1,0)$, and into 41under $(3,0)$. But these two numbers add up to $66 = 2 \times 33$, so the magic sum is obtained by averaging the sums obtained by considering two inverse actions on the original left bottom square. Some playing with the whole GMS shows that this is the case starting at any other 33-sum square and applying to it any two inverse actions. This wonderful result, together with the previous proposition, should deserve being stated as a Theorem.

Theorem 8 *Let Q be a 2×2 square window such that $S(Q) = 33$, and let (g, h) be any element of G_4. Then the following holds:*

- *Either the index of (g, h) is 0, and $S((g, h)(Q)) = 33$,*
- *or the index of (g, h) is 1, and the magic sum is the average*

$$\frac{1}{2}\left[S((g, h)(Q)) + S((-g, -h)(Q))\right] = 33$$

3.2 Diagonal 4-windows

The analysis of square 4-structures is easily carried over to diagonal 4-structures. In GMS we find three classes thereof, up to the obvious symmetry:

- principal

			*
		*	
	*		
*			

- broken 3/1

*		+	
	+		*
+		*	
	*		+

- broken 2/2

	*		
*			
			*
		*	

In the torus, the broken diagonals do not show up as "broken", and it is easy to see that zero index actions translate principal diagonals either into themselves or into broken 2/2 ones, while index 1 actions transform the principal diagonal into broken 3/1 ones. The analogue of theorem 8 is:

Theorem 9 *Let D be a diagonal 4- window such that $S(D) = 33$, and let (g, h) be any element of G_4. Then the following holds:*

- *Either the index of (g, h) is 0, and $S((g, h)(D)) = 33$,*
- *or the index of (g, h) is 1, and the magic sum is the average*

$$\frac{1}{2}\left[S((g, h)(D)) + S((-g, -h)(D))\right] = 33$$

4 Conclusion

We have presented from a heuristic viewpoint an elementary study of the unusual GMS, trying to emphasise the role of more or less hidden symmetries in preserving the magic sum. Thus we were led to the translation of arithmetical questions into geometrical considerations on a rather complicated though natural surface, that of a torus, a fact that should emphasise interest on the consideration of the unity of Mathematics and the interplay between different techniques as the cornerstone of mathematical activity.

In this paper no attempt has been made either towards extremely rigourous theorem proving or precise definitions. Rather, playing with figures and diagrams, scribbling and –to say it in a single word– discovery have been mainly stressed. For instance, the word "window" has been used interchangeably to describe both the window itself and the set of numbers seen through it, and it is obvious that Theorems 8 and 9 could be stated jointly in a single result, that could have been rewritten in terms of subgroups of G_4.The reader is invited to do it as an exercise in mathematical elegance.

References

[1] Andrews W (1917) *Magic Squares and Cubes* (Dover edition, 1960).

[2] Cirlot J (2002) *Gaudí: An Introduction to his Architecture*, Triangle-Postals, Barcelona.

[3] Maritz P (2001) The Magic Square on Sagrada Famìlia, *The Mathematical Intelligencer*, 23(4), 49-54.