Mathematical Analysis in Spain: From Cauchy to Weierstrass during the last third of the 19th Century

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Abstract

Rigorous Mathematical Analysis in the Cauchy style was not easily accepted within the European mathematical community. In average, only about forty years after the \textit{Cours d’Analyse} was published in 1821, the viewpoint of Cauchy became well established in the more mathematically advanced countries, a paradigm that was in use until the formalism based on the arithmetisation of Analysis due to Weierstrass replaced at the end of the 19\textsuperscript{th} century. In this paper the authors show how the conception of Mathematical Analysis in Spain adopted the Cauchy viewpoint quite late, around 1880, and then switched to the Weierstrassian standpoint in less than forty years, between 1880 and 1915.

1 Introduction

It is known that rigour in Mathematical Analysis was introduced in Spain by Julio Rey Pastor (1888-1962) after some stages in Germany, when he published the two basic books \textit{Elementos de Análisis Algebraico} (Rey Pastor, 1917) and \textit{Teoría de las Funciones Reales} (Rey Pastor, 1918), based on the theories by Cauchy, Weierstrass, Cantor, and Dedekind.

Nevertheless, before Rey’s time there were several attempts to introduce the rigour in the presentation of Mathematical Analysis in the Spanish educational...
system. A description and critique of these attempts is the aim of the present paper.

According to Belhoste (Belhoste, 1991) the viewpoints of Cauchy on Mathematical Analysis were not accepted in a straightforward manner, neither in France nor in any other country. Nevertheless, following Cauchy’s exile in 1830, his followers Navier, Sturm, Liouville and Duhamel kept the ideas of Cauchy and used them in teaching and mathematical writing, although with less accuracy and sometimes mixing them up with other mathematical traditions (see e.g. (Grattan-Guinness, 2000, p.67)). The last cited author, Duhamel, became, through his Cours d’Analyse, one of the forerunners of the introduction of the Cauchy style in Spain, an event that took place around 1880.

The second step towards rigour, the Weierstrassian revolution, was accepted in France when Camille Jordan (1838-1922) published his own Cours d’Analyse de l’École Polytechnique (Jordan, 1893-96) where the $\epsilon – \delta$ style was adopted. Nevertheless, the older Cauchy style was in use for several years, Lützen points out that the text by Sturm was being used in Copenhagen as late as in 1915 (Lützen, 2003).

In the middle of the 19th century there was a gradual recovery of the Spanish University system after the long reign of Fernando VII. The basic works of the mathematical scene were translated and/or adapted into Spanish and thus the newer ideas of Gauss, Cauchy and Abel in Mathematical Analysis, the birth of non-euclidean geometries, projective geometry and the modern Funktionentheorie according to Riemann became available to the Spanish mathematical community. At first, the books available for translation / adaptation did not yet include the ideas by Cauchy. In (Pacheco Castelao et al., 2005) the authors showed how the texts by Vallejo and Feliu introduced notions such as limit, function, infinitesimal, etc., but did not use these term in their proofs. Grabiner points out that the insistence in proof is a characteristic feature of the development and foundations of Analysis in the Cauchy style (Grabiner, 1981).

1880 was the year when the older pre-Cauchy Analysis disappeared from Spanish higher education. Two mathematicians, Archilla and Portuondo, wrote the books where the Cauchy vision of Analysis was, for the first time, presented in Spanish words. They elaborate on the new ideas, although always through the work of Duhamel.
Simón Archilla (1836-1890) was a civil engineer who taught Mathematics at the Universities of Barcelona and Madrid. His courses comprised Higher Algebra, Analytic Geometry, and Differential and Integral Calculus. His book *Principios del Cálculo Diferencial* (The Principles of Differential Calculus, Principios from now on) appeared in 1880, though in this paper a second edition of 1894 -completed by his son Faustino Archilla after his father’s notes-will be used (Archilla y Espejo, 1894).

Archilla was elected to the Real Academia de Ciencias Exactas, Físicas y Naturales, (The Royal Academy for the Exact, Physical, and Natural Sciences) and his inaugural dissertation, read on June 10th 1888 had the title “*On the concept and fundamental principles of Infinitesimal Calculus*” (Archilla y Espejo and Vicuña y Lazcano, 1888) where he made a more interesting historical report on the idea of infinity, from Archimedes to the date of the discourse. In page 61 of the report, Archilla acknowledged the role of Cauchy¹.
there was a need to achieve what seemed so difficult from the very beginning: To force the notion of infinity in order to serve the needs of Analysis. Opposite to this was not only the special consideration of infinity in mathematical considerations as applications to number and distance, but as well the philosophical criterion serving as a basis to the legitimate intervention of infinity in Analysis.

Moreover, he points out how \(^2\) “under the light of the new doctrine, it has become common knowledge that the ratio of certain infinitesimally small amounts, although always finite, does not tend to any limit whatsoever; it is clearly seen that the continuity of a function does not imply that of its increment and that of the variable be of the same order; and it is possible to conceive and to determine, as Weierstrass has shown, continuous functions without derivative at any point: Such things, if not unconceivable, were difficult to understand and to explain with the old ideas”.

In the foreword to Principios the author shows the basis of the discourse of his book by a quote obtained from the preface of Hoüel’s Cours de Calcul Infinitésimal, that had been translated into Spanish in 1878:

There is only a rigorous method to present Infinitesimal Calculus: It is the method of the infinitely small, or of the limits, the method of Cauchy and Duhamel...

This opinion is clearly detailed in the introduction to Principios \(^3\):

Our aim in this book is to summarize the most important principles of Infinitesimal Calculus, trying to explain their natural interrelations and to study the intimate relationships between the fundamental notions upon which they are based and those that are legitimately followed from them, according to the doctrine first expounded by Cauchy and, then, developed by Duhamel...

Finally, the difference between this text and its predecessors is featured by the treatment given to the ideas of continuity and differentiability in the light of infinitesimals (Introduction, page VI): “In the study of functions we have focus on the notion of continuity, by showing the difficulties of studying it, directly, and by relating it indirectly with the idea of infinitesimal quantities through the notion of limit…”

2.1 Archilla on infinitesimals

Archilla works with infinitesimals as the fundamental notion in this work. Thus, after introducing the concept of variable in page 1: A variable quantity, or simply a variable, is a quantity that can take a series of successive values according to any prescribed law. He goes on by explaining the manners of operating with variables according to these successive values determining them, and afterwards he defines infinitely small and infinitely large quantities in page
4: A variable quantity that can take values smaller than any given quantity and can, indefinitely, satisfy this condition is an infinitely small quantity, or simply an infinitely small. On the other hand, if a variable quantity can take values larger than any given quantity and can indefinitely satisfy this condition is an infinitely large quantity, or simply an infinitely large. Finally, constant quantities and quantities that are neither infinitely small nor infinitely large are called finite quantities.

The definition of limit is presented in a way that directly matches the definition of an infinitesimally small: The limit of a variable quantity is a constant quantity such that the variable one approaches it in such a way that the difference between the constant and the successive values of the variable becomes smaller than any given quantity, but never equal to zero.

Therefore, from this definition it follows that:

1. The difference between a variable and its limit is an infinitesimally small.
2. Every infinitesimally small has zero limit.
3. Every infinitely large has no limit.
4. The limit of a variable is a constant that can not be found among the successive values of the variable.

The last observation, directly inherited from Cauchy, introduces a certain lack of generality, for sequences tending to zero like

\[ 1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, \ldots \]

would not comply with such a definition. Nevertheless, this observation can be considered as a minor flaw. More interesting is the idea of an extended real line expounded in this paragraph obtained from page 6: “Our aim has been to give more generality to this doctrine by encompassing in a common classification finite quantities, infinitely small and infinitely large ones, thus establishing a general theory of the orders of quantity.”

Then Archilla proceeds into the definition of infinitely small, or large, or constant quantities via the ratios between two quantities:

When the ratio \( \frac{x}{y} \) is an infinitely small \( \alpha \), then \( x \) is said to be infinitely small relative to \( y \).

When the ratio \( \frac{x}{y} \) is an infinitely large \( A \), then \( x \) is said to be infinitely large relative to \( y \).

Finally, when the ratio \( \frac{x}{y} \) is some constant \( a \), then \( x \) is said to be of the same order of \( y \).

And he goes on by specifying a unit for comparison: “Once a fundamental
infinitesimally small $\alpha$ has been arbitrarily chosen, we shall consider every
infinitesimal of the same order as a first-order infinitesimal”.

Thus, if $\beta$ is any other first order infinitesimal, then some finite quantity $a$ will
exist such that $\frac{\beta}{\alpha} = a + \omega$, where $\omega$ is an infinitesimally small, from where

$$\beta = \alpha(a + \omega).$$

And in the same way the definition of an infinitesimal of $n$-th order with
respect to the basic one is obtained:

$$\beta_n = \alpha^n(a + \omega).$$

Moreover, by acknowledging that $A = \frac{1}{\alpha}$ is an infinitely large, the orders of the
infinitely large are defined likewise through the following chain of equalities:

$$B_n = A^n(a + \omega) = \left(\frac{1}{\alpha}\right)^n (a + \omega) = \alpha^{-n}(a + \omega)$$

so the whole scale of infinitesimals, ranging from the infinitely small to the
infinitely large through finite quantities is obtained by allowing the exponent
$n$ to run over the real numbers.

In the above computations there is an implicit assumption: The product of
an infinitely small $\omega$ and a finite quantity $a$ is, again, an infinitely small.
The proof of this fact is found in page 15, with some reminiscences of the
Weierstrassian style: In order to prove it, it suffices to show that $a\omega < \frac{1}{\delta}$, where
$\frac{1}{\delta}$ is an arbitrarily small quantity. But this inequality can be also written in
the form $\frac{1}{\omega} > a\delta$, and since $\omega$ is an infinitesimally small, its inverse is larger
than any given number. Therefore $a\omega$ is an infinitely small quantity.

With the above remarks Archilla presents the usual algebraic rules for in-
finitesimals and settles the relationships between limits and infinitesimals by
noticing (p. 20) that “if the difference between a variable $x$ and a constant
$a$ is infinitely small, this constant is the limit of the variable. Therefore the
equation

$$x = a + \omega$$

necessarily implies that $\lim x = a$”.

2.2 Archilla on continuous functions

The notion of continuity is presented by Archilla in two stages. In the first
one a general idea of continuity, for any variable quantity, is presented (page
A variable quantity \( x \) varies in a continuous manner if it necessarily takes every intermediate value when going from \( a \) to \( b \), and moreover the same property is true for any subinterval \([a_1, b_1] \subset [a, b]\). This definition has a geometric flavour, and we find it more interesting (with respect to the rigour of the definition) the precision made about the restriction to any subinterval.

In a second stage the notion of continuous function is introduced (page 57): “A function of a variable \( x \) is said to be continuous between the values \( x = a \) and \( x = b \) when \( x \) varies in a continuous way between these values, if the values of the function can not go from any value \( m \) to another \( n \), both between \( f(a) \) and \( f(b) \), without passing through all intermediate values between \( m \) and \( n \).”

This is the definition that Medvedev (Medvedev, 1991, p. 60-62) called “continuity in the sense of Dirichlet-Lobachevskii”, and obviously needs one more condition, i.e. that of monotonicity of the function, which is implicitly assumed, as one can observe in the figure of page 58 from the definition of discontinuity.

Nevertheless, when arriving to practical questions, Archilla does not forget his programme on infinitesimals. In page 58 he recalls: “A function is continuous on a given interval if, to any infinitely small increment of the variable in that interval, there corresponds an infinitely small increment of the function”. Next we see how the continuity of the logarithm is assessed:

Let the increment of the logarithm be \( \log(x + h) - \log(x) = \log\left(\frac{x+h}{x}\right) \). Now we know -because it had been proved earlier in the book- that, for any fixed \( x \), \( \log\left(1 + \frac{h}{x}\right) \) and \( \frac{h}{x} \) differ by an infinitely small \( \omega \) of order larger than \( \frac{h}{x} \). Therefore we see that the increment is an infinitely small, namely

\[
\log\left(1 + \frac{h}{x}\right) = \log(x + h) - \log(x) = \frac{h}{x} + \omega.
\]

Therefore we see that Archilla considers that the Cauchy viewpoint on continuous functions and the Dirichlet-Lobachevskii one are equivalent, without taking care of proving it: To him, that was simply true.

2.3 Archilla on derivable functions and on differentials

Differential calculus deals on how to establish infinitesimal relationships between the increments of the independent variable(s) and the dependent vari-

* The modern notation is ours
able, or function. Archilla denotes those increments by $\Delta x$ and $\Delta \varphi(x)$, and he writes (pages 65-66): “the direct consideration of the limit of $\frac{\Delta \varphi(x)}{\Delta x}$ for $\Delta x$ tending to zero greatly simplifies the research whose aim is to determine the infinitesimal relationships between $\Delta x$ and $\Delta \varphi(x)$, and it introduces in the Analysis one of its most important objects. The limit is called the derivative or derived function of $\varphi(x)$”.

To show how Principios is still a mixture of old and new ideas, notice that Archilla sticks to the old idea that the variable cannot take the value of the limit when approaching it and, on the other hand, he insists on a function being continuous for it to be differentiable, acknowledging that this condition is not sufficient. He explains this by considering the infinitesimal orders of the two increments.

In the last remark, we realise that, in Principios, there are no references to either Weierstrass or the existence of continuous functions without derivative anywhere. Nevertheless, Archilla knew about these functions after the first edition of Principios, since he spoke about them in his 1888 discourse for the Academia, thus the conclusion is that he did not find it necessary to include such theories in a possible second edition, which appeared after his death in 1894.

The treatment of the mean value theorems shows clearly the influence of Cauchy. In a very complex paragraph in page 76, we read that if the derivative $\varphi'(x)$ is a continuous function for all values of $x$ between $x_0$ and $x_0 + h$, then its values will be finite ones, and their mean value $\mu(\varphi'(x_h))$ will be some value between the maximum and the minimum of the derivative on that interval. Therefore, due to the assumed continuity of the derivative there exists a value $x_d$ such that $\varphi'(x_d) = \mu(\varphi'(x_h))$ and the equation (he has obtained it previously as an “interesting property” of the derivative)

$$\varphi(x_0 + h) - \varphi(x_0) = h\varphi'(x_h)$$

can be rewritten as

$$\varphi(x_0 + h) - \varphi(x_0) = h\varphi'(x_d) = h\varphi'(x + \theta h)$$

from which follows a completely modern version of Rolle’s Theorem.

It must be notice that this interesting property is simply an integral-free version of the mean value for integrals as applied to the continuous derivative

$$\varphi(x_0 + h) - \varphi(x_0) = \int_{x_0}^{x_0+h} \varphi'(s)ds = h\varphi'(x_d)$$

that the author obtained through a rather involved argument in pages 74 and 75.
The largest part of Principios extends from page 115 until the end of the book, under the title “Book III: Differential Calculus”, and it starts with the concept of the differential of a function.

Let us read:

When studying the form and general properties of the infinitely small increment of functions of one or several variable, we realised that among the infinite series of all quantities that differ infinitely little from the increment of the function, there existed one simpler than any other quantity, and it was expressed through the derivative or partial derivatives of the function; moreover, we saw that this unique form, in the case of a function of a single variable \( y = \varphi(x) \), was

\[
\varphi'(x) \Delta x
\]

This infinitely small quantity, which is unique by his form among all quantities that differ infinitely little from \( \Delta y \), will be called the differential of \( y = \varphi(x) \) and will be represented by the special notation \( dy \) or \( d\varphi(x) \)

This surprisingly modern definition lacks only the words “linear function” to be a fully current one. Nevertheless, Archilla is well aware of the linearity of the differential as a function of the increment of the independent variable:

The various values taken by the differential \( d\varphi(x) \) for a definite value of \( x \) are proportional to those of \( \Delta x = dx \); and this is a property exclusive to the differential among all quantities infinitely close to the increment \( \Delta y \)

Principios finishes without entering into the Integral Calculus. We do not know whether the author or his son Faustino Archilla intended to publish a sequel on this topic, but the fact is that no evidence has been found to support this view. Next we concentrate on three more authors which represent the history of the change of paradigm in Spanish Mathematical Analysis.

3 Clariana and his lecture notes on Mathematical Analysis: Complex Analysis

Lauro Clariana (1842-1916) was a professor (an academician) who spent his life in Barcelona and in the nearby province of Tarragona. He had studied in order to become an industrial engineer and he followed courses in Mathematics as well, the so called “Ciencias Exactas”, the name given in Spain to Mathematics for more than a century. He taught Integral and Differential Calculus, and Rational Mechanics, and was one of the few Spaniards who participated in several International Congresses and Meetings. We find him in Paris, Brussels, München, and Freiburg between 1888 and 1900, and he was awarded a prize.
for his Memoir “On the spirit of Mathematics in the modern times” in Paris in the year 1888.

Clariana was the author of two books on Mathematical Analysis for the use of students at the Escuela Superior de Ingeniería Industrial in Barcelona. They were published in 1892 and 1893 under the respective titles Resumen de las lecciones de Cálculo Diferencial e Integral (Clariana Ricart, 1892b) (Resumen from now on), and Complemento a los elementos de los Cálculos (Clariana Ricart, 1892a) (Complemento in what follows). They appeared as lithographed handwritten lecture notes, and it appears that no other editions followed these first ones. Resumen is a general introduction to Analysis, and at the very beginning Clariana makes clear his position (p.5)⁵:

Because the indefinitely small and the indefinitely large are the only elements that can become the basis of quantity in Mathematics, synthesised in the finite ones, we shall admit three categories of quantity under the following forms:

1. That of the indefinitely small.
2. That of the finite.
3. That of the indefinitely large.

And these are the only true concepts of quantity that are directly connected to the idea of differential in Leibnitz.

Then follows a particularly long introduction (Prolegómenos) of 45 pages on the various classes of numbers and functions, as well as on the foundations and history of the infinitesimal method, where he summons Descartes, Johann Bernouilli and Cournot, and indeed Newton, Leibnitz, and D’Alembert.

The rest of the book is a classical treatise of the usual topics on Differential and Integral Calculus presented in a straightforward way. Theorems are not highlighted and proofs are not distinctly offered. The main emphasis is given on the sequence of useful and applicable formulas, with a few examples spread along the text. This makes the book very readable and surprisingly modern to today’s standards.

Complemento has a different flavour. It is a compilation of loosely knit topics of higher Analysis. Clariana declares in a brief introduction:

The aim of this book is to present what we should call “modern theories of the infinitesimal and integral calculus”, not because some of them be recent ones, but because they have not yet been presented in the Spanish education.†

The first two chapters are devoted to “Infinitesimally Small Triangles” and “Orders of Comparison for Curves”, where the fundamental Leibnitzian triangle is explained and applied in depth, as well as the idea of the order of an infinitesimal is deeply applied to the study of different elements of curves.

† Our underline.
With this equipment, the book follows with a study of Classical Differential Geometry.

On the remaining chapters of Complemento a variety of topics is included. We find the Euler-McLaurin summation formula, special functions and elliptic integrals ... To summarize, it is a simplified version of the second volume of the classical French treatises that clearly inspired the author. The style is practically the same as in Resumen.

In basic questions Clariana holds the same opinions of Archilla. As an example we quote the definition of a continuous function on an interval:

The function \( y = F(x) \) is continuous between the values \( a \) and \( b \) attributed to \( x \) if it passes from one value to another through values that differ between them as little as desired.

But the main novelty in Clariana’s books is that he is the first to introduce Complex Analysis in print in Spain. He plainly states that “a complex quantity has the form \( x + yi \) where \( x \) and \( y \) are real quantities” and goes on by explaining that Gauss was the first to speak of \( yi \) as imaginary and that Cauchy denoted the whole complex quantity as imaginary. Of course he points out that when \( x \) and \( y \) are variable quantities, then \( x + yi \) is a complex variable and that a complex quantity is an infinitesimally small (resp. large) if the real part \( x \) and the imaginary part \( y \) are infinitesimally small (large) quantities. Continuity of a complex quantity is, of course, assessed form the continuity of its component real variables. Then, a standard theory follows.

It must be noted that, before Clariana, no Spanish mathematician had studied complex quantities as the object of Analysis. Only algebraic, geometric or arithmetic considerations had been made in Spain on these numbers, and for nearly forty years the source book was the rather obscure Teoría transcendental de las cantidades imaginarias, the posthumous work (1865) of José María Rey Heredia (1818-1861) who inspired several developments, especially in the presentation of Analytic Geometry. The work of Rey Heredia has been also studied elsewhere by the authors (Pacheco Castelao et al., 2006).

4 Villafañe and the Tratado de Análisis Matemático

José María Villafañe (1830-1915) was of Cuban origin and taught Mathematics at different Universities: Barcelona, Valencia, and Madrid. In 1892 he published a treatise on Mathematical Analysis (Villafañe y Viñals, 1892) in three parts, which he enlarged afterwards to four parts. The first one deals with the basic techniques of Analysis: Functions, continued fractions, congru-
quences, complex numbers... The second and most interesting of the aims of the present paper contains the Infinitesimal Analysis: limits, infinitesimals, continuous functions, derivatives, integrals, and infinite series. Equation solving and algebraic forms conform the respective contents of the third an fourth parts.

Although Villafañe’s book studies roughly the same topics presented in the books analysed before, it must be noted that its presentation is much more formal and rigorous. Here we shall dwell only in some remarkable differences. A first one is observed in the definition of a limit:\footnote{A fixed quantity will be called the limit of a variable quantity if this last one indefinitely approaches the first one until the difference between them is smaller than any arbitrarily small chosen quantity. Indeed this formulation is much closer to the familiar $\varepsilon-\delta$ definition, although the absolute value is not used. Nevertheless, a few pages onwards the absolute value is used when trying to prove that $\frac{A}{0}$ is larger than any finite quantity. In Chapter III of Part II we find the definition of continuity:\footnote{A function is continuous if to (infinitely) small values of the variable there correspond infinitely small increments -in absolute value- of the function. Villafañe does not make the distinction between continuity and uniform continuity, established by Heinrich Heine (1821-1881) in 1872 (see (Heine, 1872)), although Dirichlet had already dealt with it as early as 1854. What Villafañe is trying to do is to prove the equivalence between the definitions of Cauchy and of Dirichlet-Lobachevskii. Complex Analysis is also considered in this treatise, as it was in the books by Clariana. But Villafañe goes further by offering for the first time in a Spanish text a fairly good approximation to the correct definition of continuity of a function of a complex variable, when he writes: “the function $u = f(z)$ of the imaginary variable $z$ is continuous on the (plane) surface limited by a closed contour if for each value of $z$ within the determined bounds the modulus of the functional increment $\Delta u = f(z + \Delta z) - f(z)$ tends to zero when the modulus of $\Delta z$ tends to zero”\footnote{Moreover, holomorphic functions are defined before the derivative of a complex function as “continuous, monotropic and monogenic functions, or admitting a well determined derivative” in page 194 of Part II.}”. Moreover, holomorphic functions are defined before the derivative of a complex function as “continuous, monotropic and monogenic functions, or admitting a well determined derivative” in page 194 of Part II.

There are some bulk errors in Villafañe’s treatise, and here we consider two of them. When dealing with derivatives (Part II, page 211) we find: “There exists a finite limit for the ratio of the infinitely small increments $k$ and $h$ of a continuous variable and its independent variable, and only some exceptional values of the variable can make the ratio grow indefinitely towards infinity or indefinitely decrease towards zero”\footnote{There are some bulk errors in Villafañe’s treatise, and here we consider two of them. When dealing with derivatives (Part II, page 211) we find: “There exists a finite limit for the ratio of the infinitely small increments $k$ and $h$ of a continuous variable and its independent variable, and only some exceptional values of the variable can make the ratio grow indefinitely towards infinity or indefinitely decrease towards zero”}.
This is obviously wrong, since the statement affirms that every continuous function is a differentiable but for a set of exceptional points, and it contradicts the Hankel condensation principle. Upon usage of this method Hankel managed to present, in 1870, a construction of a continuous function without derivative at any rational point. Strangely enough, Villafañe knew, if not about this construction, at least from the 1872 example by Weierstrass of a continuous function without derivative at any point.

Nevertheless, we can read in page 216 of Part II: “We do not deem it necessary to take into account the theorem of Weierstrass where he affirms the existence of continuous functions without derivative at any point, a matter still under discussion. Even if we admitted the existence of such functions, it would not interfere what we are going to expound on the derivatives of continuous functions”.

The calculus of complex functions is presented in page 391 ff. The first result is the assertion that Cauchy conditions are necessary and sufficient conditions for a function “to have a correctly determined derivative with respect to the complex variable $z$, i.e. to be a monogenic function”. The proof is essentially the same given by Cauchy (Cauchy, 1882-1974, 1$^{a}$ Ser. T. I, p. 330) and continuity of the partial derivatives is used without having postulated it. Indeed, with this addition, Cauchy conditions become sufficient ones.

5 Pérez de Muñoz and his Elementos de Cálculo Infinitesimal

Ramón Pérez de Muñoz was a professor of Mathematics at the Escuela Superior de Ingenieros de Minas in Madrid. He taught Calculus and Mechanics, and was the author of a book published in 1914 and entitled Elementos de Cálculo Infinitesimal (Pérez de Muñoz, 1914). According to the author’s foreword, the book is inspired in the works of Archilla, Villafañe, La Vallée-Poussin, Duhamel, Cantor and others. In addition, he explains that his aim is to present a rigorous and scientifically clear exposition, even at the cost of overflowing the usual contents of books for engineers. Thus, we find several novelties in this text.

The first chapters deal with number systems, and this book is the first Spanish one presenting a construction of the real field following Cantor and Dedekind. It starts with the notion of set and that of the cardinality of a set, and it is proved that between any two (rational) numbers there are infinitely many ones. After these “cuts” in the sense of Dedekind are presented, in a rather rhetoric manner the continuity and non denumerability of the real line is established. The theorem on the existence of least upper bound for any bounded set is also presented.
Limits are defined in a modern way, although inaccessibility of the limit is still preserved. Nevertheless, the proof that a monotonic and bounded sequence has a limit is a completely rigorous one, based on the previous ideas on real numbers.

For the first time, we find the theorem asserting that any function having a derivative at some point is also continuous at that point, although the hypothesis of continuity in a neighbourhood of the point is not stated. Summarizing, there is a large qualitative leap forward in this book when compared with those by Archilla, Clariana or Villafañe.

6 Conclusions

We have shown in this paper that the work of several mathematicians and engineers must be acknowledged in the enterprise of introducing rigour in the teaching and spread of Mathematical Analysis in Spanish Universities and Engineering Schools. Here we have dealt with the most representative four personalities. Next we highlight some of their achievements:

(1) Archilla is the first Spanish mathematician to introduce the Cauchy style in his book *Principios de Cálculo Diferencial*.
(2) Clariana was the first to present complex analysis, as well as the definition of continuity of a complex function.
(3) Villafañe made an interesting effort by presenting, for instance, *e.g.*
   - The total derivative of a complex function.
   - Several definitions related with complex functions, among them that of a holomorphic function.
   - The relationship between the existence of a derivative and the Cauchy-Riemann conditions for continuous functions.
   - Absolute values as a tool in proofs and definitions.
(4) Pérez de Muñoz presents, for the first time, the construction of the real field and the proof of the continuity of a derivable function.

Indeed, the texts by these authors are not milestones of the road to mathematical knowledge, but they must be acknowledged since they represent the steps towards introducing Spain in the mainstream of Mathematical Analysis during the first two decades of the 20th century.

Notes
era necesario conseguir lo que desde un principio tan difícil aparecía: había necesidad de obligar la noción del infinito a someterse dócil y servir de instrumento a las necesidades del análisis; y a esto se oponía no sólo el carácter y sello especial con que la noción del infinito se mostraba a la consideración matemática, cuando se aplicaba al número y a la distancia, sino también el criterio filosófico que había de servir de fundamento a la legítima intervención del infinito en el análisis.

a la luz de la nueva doctrina, es ya noción vulgar la de la razón de ciertos infinitamente pequeños, que, aunque siempre finita, no tiende a límite alguno; se ve claramente que la continuidad de una función no implica que los incrementos de ésta y de su variable hayan de ser del mismo orden; y se conciben y determinan, como lo ha hecho Weierstrass, funciones continuas que no tienen derivada: cosas, si no inconcebibles, difíciles de entender y de explicar en el antiguo orden de ideas.

Nos proponemos resumir en este libro los principios más importantes del Cálculo diferencial, procurando establecer su natural subordinación y dependencia, y estudiar las íntimas relaciones que existen entre las nociones fundamentales que les sirven de base y las que de éstas legítimamente se derivan, conforme a la doctrina propuesta primero por Cauchy, y desarrollada después por Duhamel.

Si la derivada \( \varphi'(x) \) de la función de que se trata es también continua para todos los valores de \( x \) comprendidos entre \( x_0 \) y \( x_0 + h \), los valores de la derivada serán finitos en este intervalo, y su media \( \mathcal{M}(\varphi'(x_h)) \) tendrá siempre un valor comprendido entre el mayor y menor de dichos valores de la derivada; por consiguiente, hay un valor \( x_d \) de la variable para el cual la derivada \( \varphi'(x) \) es igual a dicha media, y la ecuación \( \varphi(x_0 + h) - \varphi(x_0) = h\mathcal{M}(\varphi'(x_d)) \) puede escribirse con otra forma

\[
\begin{equation}
(B) \quad \varphi(x_0 + h) - \varphi(x_0) = h\varphi'(x_d);
\end{equation}
\]

y como \( x_d \) es uno de los valores de la variable comprendidos entre \( x_0 \) y \( x + h \), será

\[
x_d = x_0 + \theta h,
\]

en cuya expresión es \( \theta \) un número positivo, en general comprendido entre cero y la unidad, que también en casos particulares podrá tomar uno de estos dos valores. La ecuación \( (B) \) puede, por lo tanto, escribirse como sigue:

\[
\begin{equation}
(C) \quad \varphi(x_0 + h) - \varphi(x_0) = h\varphi'(x_0 + \theta h).
\end{equation}
\]

Siendo los indefinidamente pequeños, así como los indefinidamente grandes, lo únicos elementos que pueden constituir la base de la cantidad en matemáticas, sintetizados en lo finito, admitiremos en esta ciencia tres categorías de cantidad, bajo la forma siguiente:

1. Correspondiente a los indefinidamente pequeños.
2. Correspondiente a lo finito.
3. Correspondiente a los indefinidamente grandes.
Estos son los únicos y verdaderos conceptos de cantidad que se enlazan directamente con la diferencial de Leibnitz.

6Se llama límite de una variable la cantidad fija, a que esta variable se aproxima indefinidamente hasta poder ser la diferencia entre la variable y la cantidad fija menor que toda magnitud tan pequeña como se quiera.

7Una función es continua, cuando para incrementos pequeños de la variable, los incrementos correspondientes de de la función son también infinitamente pequeños en valor absoluto.

References

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