A RADON-NIKODYM THEOREM FOR COMPLETELY MULTI-POSITIVE LINEAR MAPS AND ITS APPLICATIONS

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March 29, 2006

Abstract

A completely $n$-positive linear map from a locally $C^*$-algebra $A$ to another locally $C^*$-algebra $B$ is an $n \times n$ matrix whose elements are continuous linear maps from $A$ to $B$ and which verifies the condition of completely positivity. In this paper we prove a Radon-Nikodym type theorem for strict completely $n$-positive linear maps which describes the order relation on the set of all strict completely $n$-positive linear maps from a locally $C^*$-algebra $A$ to a $C^*$-algebra $B$, in terms of a self-dual Hilbert $C^*$-module structure induced by each strict completely $n$-positive linear map. As applications of this result we characterize the pure completely $n$-positive linear maps from $A$ to $B$ and the extreme elements in the set of all identity preserving completely $n$-positive linear maps from $A$ to $B$. Also we determine a certain class of extreme elements in the set of all identity preserving completely positive linear maps from $A$ to $M_n(B)$.

MSC: 46L05; 46L08

1 Introduction and preliminaries

The concept of matricial order plays an important role to understand the infinite-dimensional non-commutative structure of operator algebras. Completely positive maps, as the natural ordering attached to this structure have been studied extensively [1, 2, 5, 4, 7, 8, 9, 10, 11, 15, 16, 18, 19].

Given a $C^*$-algebra $A$ we denote by $M_n(A)$ the $C^*$-algebra of all $n \times n$ matrices with elements in $A$.

Definition 1.1 A completely positive map from a $C^*$-algebra $A$ to another $C^*$-algebra $B$ is a linear map $\rho : A \to B$ such that the linear map $\rho_n : M_n(A) \to M_n(B)$...
$M_n(A) \rightarrow M_n(B)$ defined by

$$\rho_n \left( [a_{ij}]_{i,j=1}^n \right) = [\rho(a_{ij})]_{i,j=1}^n$$

is positive for each positive integer $n$.

In 1955, Stinespring [18] showed that a completely positive linear map $\rho$ from a $C^*$-algebra $A$ to $L(H)$, the $C^*$-algebra of all bounded linear operators on a Hilbert space $H$, induces a representation $\Phi_\rho$ of $A$ on another Hilbert space $H_\rho$. Moreover,

$$\rho(a) = V_\rho^*\Phi_\rho(a)V_\rho$$

for all $a \in A$ and for some bounded linear operator $V_\rho$ from $H$ to $H_\rho$.

In 1969, Arveson [1] proved a Radon-Nikodym type theorem which gives a description of the order relation in the set of all completely positive linear maps from $A$ to $L(H)$ in terms of the Stinespring representation associated with each completely positive linear map. The gist of the proof of this result is the fact that any bounded linear operator on a Hilbert space is adjointable.

Hilbert $C^*$-modules are generalizations of Hilbert spaces by allowing the inner-product to take values in a $C^*$-algebra rather than in the field of complex numbers.

**Definition 1.2** A pre-Hilbert $A$-module is a complex vector space $E$ which is also a right $A$-module, compatible with the complex algebra structure, equipped with an $A$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \rightarrow A$ which is $\mathbb{C}$- and $A$-linear in its second variable and satisfies the following relations:

1. $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$;
2. $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$;
3. $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$.

We say that $E$ is a Hilbert $A$-module if $E$ is complete with respect to the topology determined by the norm $\| \cdot \|$ given by $\| \xi \| = \sqrt{\langle \xi, \xi \rangle}$.

A $C^*$-algebra $A$ is a Hilbert $C^*$-module over $A$ with the inner-product defined by $\langle a, b \rangle = a^*b$ for $a$ and $b$ in $A$.

Given two Hilbert $A$-modules $E$ and $F$, the Banach space of all bounded module homomorphisms from $E$ to $F$ is denoted by $B_A(E, F)$. The subset of $B_A(E, F)$ consisting of all adjointable module homomorphisms from $E$ to $F$ (that is, $T \in B_A(E, F)$ such that there is $T^* \in B_A(F, E)$ satisfying

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle$$

for all $x \in E$ and $y \in F$) is denoted by $B^*_A(E, F)$.
\[ \langle \eta, T\xi \rangle = (T^*\eta, \xi) \] for all \( \xi \in E \) and for all \( \eta \in F \) is denoted by \( \mathcal{L}_A(E, F) \).

We will write \( \mathcal{B}_A(E) \) for \( \mathcal{B}_A(E, E) \) and \( \mathcal{L}_A(E) \) for \( \mathcal{L}_A(E, E) \).

In general, \( \mathcal{L}_A(E, F) \neq \mathcal{B}_A(E, F) \). So the theory of Hilbert \( C^* \)-modules is different from the theory of Hilbert spaces.

The Banach space \( E^\# \) of all bounded module homomorphisms from \( E \) to \( A \) becomes a right \( A \)-module with the action of \( A \) on \( E^\# \) defined by \( (aT)(\xi) = a^*(T^*\xi) \) for \( a \in A, T \in E^\# \) and \( \xi \in E \). We say that \( E \) is self-dual if \( E^\# = E \) as right \( A \)-modules.

If \( E \) and \( F \) are self-dual, then \( \mathcal{B}_A(E, F) = \mathcal{L}_A(E, F) \) [16, Proposition 3.4].

Suppose that \( A \) is a \( W^* \)-algebra. Then the \( A \)-valued inner-product on \( E \) extends to an \( A \)-valued inner-product on \( E^\# \) and in this way \( E^\# \) becomes a self-dual Hilbert \( A \)-module [16, Theorem 3.2]. Moreover, any bounded module homomorphism \( T \) from \( E \) to \( F \) extends uniquely to a bounded homomorphism \( \widetilde{T} \) from \( E^\# \) to \( F^\# \) [16, Proposition 3.6].

A representation of a \( C^* \)-algebra \( A \) on a Hilbert \( C^* \)-module \( E \) over a \( C^* \)-algebra \( B \) is a \( * \)-morphism \( \Phi \) from \( A \) to \( \mathcal{L}_B(E) \).

Paschke [16] showed that a completely positive map from a unital \( C^* \)-algebra \( A \) to another unital \( C^* \)-algebra \( B \) induces a representation of \( A \) on a Hilbert \( B \)-module which generalizes the GNS construction and he extended the Arveson’s results for completely positive maps from a unital \( C^* \)-algebra \( A \) to a \( W^* \)-algebra \( B \).

In 1996, Tsui [19] proved a Radon-Nikodym type theorem for completely positive maps between unital \( C^* \)-algebras and using this theorem he obtained characterizations of the pure elements and the extreme points in the set of all identity preserving completely positive maps from a unital \( C^* \)-algebra \( A \) to another unital \( C^* \)-algebra \( B \) in terms of a self-dual Hilbert module structure induced by each completely positive map. To prove these facts he used the following construction.

**Construction 1.3** ([15, 16, 19]) Let \( E \) be a Hilbert \( C^* \)-module over a \( C^* \)-algebra \( B \). The algebraic tensor product \( E \otimes_{\text{alg}} B^{**} \), where \( B^{**} \) is the enveloping \( W^* \)-algebra of \( B \), becomes a right \( B^{**} \)-module if we define \( (\xi \otimes b) c = \xi \otimes bc \), for \( \xi \in E \), and \( b, c \in B^{**} \).

The map \( [,] : (E \otimes_{\text{alg}} B^{**}) \times (E \otimes_{\text{alg}} B^{**}) \to B^{**} \) defined by

\[
\left[ \sum_{i=1}^{n} \xi_i \otimes b_i, \sum_{j=1}^{m} \eta_j \otimes c_j \right] = \sum_{i=1}^{n} \sum_{j=1}^{m} b_i^* \langle \xi_i, \eta_j \rangle c_j
\]
is a $B^{**}$-valued inner-product on $E \otimes_{\text{alg}} B^{**}$ and the quotient module $E \otimes_{\text{alg}} B^{**}/N_E$, where $N_E = \{ \zeta \in E \otimes_{\text{alg}} B^{**}; [\zeta, \zeta] = 0 \}$, becomes a pre-Hilbert $B^{**}$-module. The Hilbert $C^*$-module $E \otimes_{\text{alg}} B^{**}/N_E$ obtained by the completion of $E \otimes_{\text{alg}} B^{**}/N_E$ with respect to the norm induced by the inner product $[\cdot, \cdot]$ is called the extension of $E$ by the $C^*$-algebra $B^{**}$. Moreover, $E$ can be regarded as a $B$-submodule of $E \otimes_{\text{alg}} B^{**}/N_E$, since the map $\xi \mapsto \xi \otimes 1 + N_E$ from $E$ to $E \otimes_{\text{alg}} B^{**}/N_E$ is an isometric inclusion.

The self-dual Hilbert $B^{**}$-module $(E \otimes_{\text{alg}} B^{**}/N_E)^\#$ is denoted by $\tilde{E}$, and we can consider $E$ as embedded in $\tilde{E}$ without making distinction.

Let $T \in \mathcal{B}_B(E, F)$. For $b_1, \ldots, b_m \in B^{**}$ and $\xi_1, \ldots, \xi_m$ in $E$ we denote by $b$ the element in $(B^{**})^m$ whose components are $b_1, \ldots, b_m$, by $X$ the matrix in $M_n(B^{**})$ whose the $(i, j)$-entry is $\langle \xi_i, \xi_j \rangle$ and by $X_T$ the matrix in $M_n(B^{**})$ whose the $(i, j)$-entry is $\langle T \xi_i, T \xi_j \rangle$. By Lemma 4.2 in [14], $0 \leq X_T \leq \|T\| X$. Identifying $M_n(B^{**})$ with $L_{B^{**}}((B^{**})^n)$, we have

$$\left[ \sum_{i=1}^m T \xi_i \otimes b_i, \sum_{i=1}^m T \xi_i \otimes b_i \right] = \sum_{i,j=1}^m b_i^* \langle T \xi_i, T \xi_j \rangle b_j = \langle b, X_T b \rangle$$

$$\leq \|T\| \langle b, Xb \rangle = \|T\| \left[ \sum_{i=1}^m \xi_i \otimes b_i, \sum_{i=1}^m \xi_i \otimes b_i \right].$$

Therefore $T$ extends uniquely to a bounded module homomorphism $\widehat{T}$ from $E \otimes_{\text{alg}} B^{**}/N_E$ to $F \otimes_{\text{alg}} B^{**}/N_F$ such that

$$\widehat{T} \left( \sum_{i=1}^m \xi_i \otimes b_i \right) = \sum_{i=1}^m T \xi_i \otimes b_i$$

and by Proposition 3.6 in [16], this extends uniquely to a bounded module homomorphism $\bar{T}$ from $\tilde{E}$ to $\tilde{F}$ such that $\|T\| = \|\bar{T}\|$.  

Remark 1.4 Any element $T \in \mathcal{B}_B(E, F)$ extends uniquely to an element $\bar{T} \in \mathcal{B}_{B^{**}}(\tilde{E}, \tilde{F})$ such that $\|T\| = \|\bar{T}\|$. Moreover, $\bar{T} \bar{S} = \bar{T} \bar{S}$ for all $T \in \mathcal{B}_B(E, F)$ and $S \in \mathcal{B}_B(F, E)$, and if $T \in \mathcal{L}(E, F)$, then $\bar{T}^* = \bar{T}^*$.  

\[4\]
Remark 1.5 Let $T \in \mathcal{B}_B(E, E^\#)$. We extend $T$ to an element $\tilde{T} \in \mathcal{B}_B((E \otimes_{\text{alg}} B^{**})/N_E, \tilde{E})$ by

$$
\left[ T \left( \sum_{i=1}^{n} \xi_i \otimes b_i \right) , \sum_{j=1}^{m} \eta_i \otimes c_j \right] = \sum_{i=1}^{n} \sum_{j=1}^{m} b_i^* \left[ T \xi_i, \eta_j \right] c_j
$$

and then extend it again to an element $\tilde{T} \in \mathcal{B}_B(\tilde{E})$ such that $\|T\| = \|\tilde{T}\|$ [16, Proposition 3.6].

Remark 1.6 A representation $\Phi$ of a $C^*$-algebra $A$ on a Hilbert $C^*$-module $E$ over a $C^*$-algebra $B$ induces a representation $\tilde{\Phi}$ of $A$ on $\tilde{E}$ defined by $\tilde{\Phi}(a) = \tilde{\Phi}(a)$ for all $a \in A$.

Remark 1.7 Any completely positive linear map $\rho$ from $A$ to $B$ induces a representation $\tilde{\Phi}^\rho$ of $A$ on a self-dual Hilbert $B^{**}$-module $\tilde{E}_\rho$.

Locally $C^*$-algebras are generalizations of $C^*$-algebras. Instead of being given by a single norm, the topology on a locally $C^*$-algebra is defined by a directed family of $C^*$-seminorms. In fact a locally $C^*$-algebra is an inverse limit of $C^*$-algebras.

Definition 1.8 A locally $C^*$-algebra is a complete complex Hausdorff topological $*$-algebra $A$ whose topology is determined by its continuous $C^*$-seminorms in the sense that the net $\{a_i\}_{i \in I}$ converges to 0 if and only if the net $\{p(a_i)\}_{i \in I}$ converges to 0 for all continuous $C^*$-seminorm $p$ on $A$.

If $A$ is a locally $C^*$-algebra and $S(A)$ is the set of all continuous $C^*$-seminorms on $A$, then for each $p \in S(A)$, $A_p = A/\ker p$ is a $C^*$-algebra in the norm induced by $p$. The canonical map from $A$ onto $A_p$ is denoted by $\pi_p$ for each $p \in S(A)$. For $p, q \in S(A)$ with $q \leq p$ there is a unique morphism of $C^*$-algebras $\pi_{pq}$ from $A_p$ onto $A_q$ such that $\pi_{pq}(\pi_p(a)) = \pi_q(a)$ for all $a \in A$. Moreover, $\{A_p; \pi_{pq}\}_{p,q \in S(A), p \geq q}$ is an inverse system of $C^*$-algebras, and $A$ can be identified with $\lim_{\leftarrow p} A_p$. Clearly, any $C^*$-algebra is a locally $C^*$-algebra.

The terminology ”locally $C^*$-algebra” is due to Inoue [6]. In the literature, locally $C^*$-algebras have been given different name such as $b^*$-algebras, $m$-convex-$C^*$-algebras, $LMC^*$-algebras [3] or pro-$C^*$-algebras [17]. Such important concepts as Hilbert $C^*$-modules, adjointable operators, (completely) positive linear maps, (completely) multi-positive linear maps can be
defined with obvious modifications in the framework of locally \( C^* \)-algebras and many results from the theory of \( C^* \)-algebras are still valid. The proofs are not always straightforward. Thus, in [3] it is proved that a continuous positive functional \( \rho \) on a locally \( C^* \)-algebra \( A \) induces a representation of \( A \) on a Hilbert space \( H \) which extends the GNS construction, and moreover, the representation of \( A \) induced by \( \rho \) is irreducible if and only if \( \rho \) is pure. In [2], Bhatt and Karia extend the Stinespring construction for completely positive linear maps from a locally \( C^* \)-algebra \( A \) to \( L(H) \).

If \( A \) is a locally \( C^* \)-algebra, then the set \( M_n(A) \) of all \( n \times n \) matrices over \( A \) with the algebraic operations and the topology obtained byreplying it as a direct sum of \( n^2 \) copies of \( A \) is a locally \( C^* \)-algebra.

**Definition 1.9** ([4], [9]). A completely \( n \)-positive map from a locally \( C^* \)-algebra \( A \) to another locally \( C^* \)-algebra \( B \) is an \( n \times n \) matrix \( [\rho_{ij}]_{i,j=1}^n \) whose elements are continuous linear maps from \( A \) to \( B \) such that the map \( \rho \) from \( M_n(A) \) to \( M_n(B) \) defined by

\[
\rho \left( [a_{ij}]_{i,j=1}^n \right) = [\rho_{ij} (a_{ij})]_{i,j=1}^n
\]

is completely positive.

**Definition 1.10** ([9]). A completely \( n \)-positive map \( [\rho_{ij}]_{i,j=1}^n \) from \( A \) to \( L_B(E) \), where \( E \) is a Hilbert module over a \( C^* \)-algebra \( B \) is strict if for some approximate unit \( \{e_\lambda\}_{\lambda \in \Lambda} \) for \( A \), the nets \( \{\rho_{ii} (e_\lambda)\}_{\lambda \in \Lambda}, i \in \{1, ..., n\} \) are strictly Cauchy in \( L_B(E) \) (that is, the nets \( \{\rho_{ii} (e_\lambda) \xi\}_{\lambda \in \Lambda}, i \in \{1, ..., n\} \) are Cauchy in \( E \) for each \( \xi \in E \)).

**Remark 1.11** If \( A \) is unital or \( E \) is a Hilbert space, then any completely \( n \)-positive map from \( A \) to \( L(E) \) is strict.

In [9], we extend the KSGNS (Kasparov, Stinespring, Gel’fand, Naimark, Segal) construction for strict completely multi-positive linear maps between locally \( C^* \)-algebras.

**Theorem 1** ([9]). Let \( A \) be a locally \( C^* \)-algebras, let \( E \) be a Hilbert module over a \( C^* \)-algebra \( B \) and let \( \rho = [\rho_{ij}]_{i,j=1}^n \) be a strict completely \( n \)-positive map from \( A \) to \( L_B(E) \).

1. There is a representation \( \Phi_\rho \) of \( A \) on a Hilbert \( B \)-module \( E_\rho \) and there are \( n \) elements \( V_{\rho,1}, ..., V_{\rho,n} \) in \( L_B(E_\rho, E_\rho) \) such that
(a) $\rho_{ij}(a) = V^*_{\rho,i} \Phi_{\rho}(a) V_{\rho,j}$ for all $a \in A$ and for all $i, j \in \{1, \ldots, n\}$;
(b) $\{\Phi_{\rho}(a)V_{\rho,i}x; a \in A, x \in E, 1 \leq i \leq n\}$ spans a dense subspace of $E_{\rho}$.

2. If $\Phi$ is another representation of $A$ on a Hilbert $B$-module $F$ and $W_1, \ldots, W_n$ are $n$ elements in $L_B(E, F)$ such that
(a) $\rho_{ij}(a) = W^*_i \Phi(a) W_j$ for all $a \in A$ and for all $i, j \in \{1, \ldots, n\}$;
(b) $\{\Phi(a)W_i x; a \in A, x \in E, 1 \leq i \leq n\}$ spans a dense subspace of $F$,
there is a unitary operator $U \in L_B(E_{\rho}, F)$ such that
i. $\Phi(a)U = U\Phi_{\rho}(a)$ for all $a \in A$; and
ii. $W_i = UV_{\rho,i}$ for all $i \in \{1, \ldots, n\}$.

The $n + 2$ tuple $(\Phi_{\rho}, E_{\rho}, V_{\rho,1}, \ldots, V_{\rho,n})$ is called the KSGNS construction associated with $\rho$.

In [10], we prove a Radon-Nikodym type theorem for completely multi-positive linear maps from a locally $C^*$-algebra $A$ to $L(H)$ and we characterize the pure elements and the extreme points in the set of all identity preserving completely multi-positive linear maps from $A$ to $L(H)$ in terms of the representation of $A$ induced by each completely multi-positive linear map. Also, we determine a certain class of extreme points in the set of all identity preserving completely positive linear maps from $A$ to $M_n(L(H))$. In this talk, we will extend the results from [10] for completely multi-positive linear maps from a locally $C^*$-algebra $A$ to a $C^*$-algebra $B$.

2 The Radon-Nikodym theorem for completely $n$-positive linear maps

Throughout this section, we assume that $A$ is a locally $C^*$-algebra, $B$ is a $C^*$-algebra and $E$ is a Hilbert $C^*$-module over $B$. We will denote by $SCP^n_\infty(A, L_B(E))$ the set of all strict completely $n$-positive linear maps from $A$ to $L_B(E)$ and by $CP^n_\infty(A, L_B(E))$ the set of all completely positive linear maps from $A$ to $L_B(E)$.

**Proposition 2.1** ([4],[10]) There is a bijection $S$ from the set $CP^n_\infty(A, B)$ of all completely $n$-positive maps from $A$ to $B$ onto the set $CP^n_\infty(A, M_n(B))$ of all completely positive maps from $A$ to $M_n(B)$ defined by

$$S\left( [\rho_{ij}]_{i,j=1}^n \right)(a) = [\rho_{ij}(a)]_{i,j=1}^n$$

for all $a \in A$. 

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which preserves the order relation.

For an element \( T \in \mathcal{L}_{B^{**}}(E) \) we denotes by \( T|_E \) the restriction of the map \( T \) on \( E \).

Let \( \rho \in SCP^\infty(A, \mathcal{L}_B(E)) \). We denote by \( C(\rho) \) the \( C^* \)-subalgebra of \( \mathcal{L}_{B^{**}}(E_\rho) \) generated by \( \{ T \in \mathcal{L}_{B^{**}}(E_\rho); T\widetilde{\Phi}_\rho(a) = \widetilde{\Phi}_\rho(a) T, \widetilde{V}_{\rho,j}^* T \Phi_\rho(a) \widetilde{V}_{\rho,i} \mid_E \in \mathcal{L}_B(E) \text{ for all } a \in A \text{ and for all } i, j \in \{1, \ldots, n\} \} \).

**Remark 2.2** If \( T \) is an element in \( C(\rho) \), then \( T|_{E_\rho} \in \mathcal{B}_B(E_\rho, E_\rho^*) \), since

\[
\langle T\Phi_\rho(a) V_{\rho,j} \xi, \Phi_\rho(b) V_{\rho,i} \eta \rangle = \langle T\widetilde{\Phi}_\rho(a) \widetilde{V}_{\rho,j} \xi, \widetilde{\Phi}_\rho(b) \widetilde{V}_{\rho,i} \eta \rangle \in B
\]

for all \( a, b \in A \), for all \( \xi, \eta \in E \) and for all \( i, j \in \{1, \ldots, n\} \) and since \( \{ \Phi_\rho(a) V_{\rho,i} \xi; a \in A, \xi \in E, 1 \leq i \leq n \} \) spans a dense submodule of \( E_\rho \).

**Lemma 2.3** Let \( T \in C(\rho) \). If \( T \) is positive, then the map \( \rho_T \) from \( M_n(A) \) to \( M_n(\mathcal{L}_B(E)) \) defined by

\[
\rho_T(\begin{bmatrix} a_{ij} \end{bmatrix}_{i,j=1}^n) = \begin{bmatrix} \widetilde{V}_{\rho,i} T \widetilde{\Phi}_\rho(a_{ij}) \widetilde{V}_{\rho,j} \end{bmatrix}_{i,j=1}^n
\]

is a strict completely \( n \)-positive linear map from \( A \) to \( \mathcal{L}_B(E) \).

**Proof.** It is not difficult to see that \( \rho_T \) is an \( n \times n \) matrix of continuous linear maps from \( A \) to \( \mathcal{L}_B(E) \), the \((i,j)\)-entry of the matrix \( \rho_T \) is the linear map \( (\rho_T)_{ij} \) from \( A \) to \( \mathcal{L}_B(E) \) defined by

\[
(\rho_T)_{ij}(a) = \widetilde{V}_{\rho,i} T \widetilde{\Phi}_\rho(a_{ij}) \widetilde{V}_{\rho,j} \mid_E.
\]

Also it is not difficult to check that for all \( a_1, \ldots, a_m \in A \) and for all \( T_1, \ldots, T_m \in M_n(\mathcal{L}_B(E)) \), we have

\[
\sum_{k,l=1}^m T_i^* S(\rho_T) (a_i^* a_k) T_k = \left( \sum_{k,l=1}^m \widetilde{T}_i^* S(\rho_T) (a_i^* a_k) \widetilde{T}_k \right) \bigg|_E = \left( \sum_{l=1}^m M_{T_{\frac{1}{2}}} (a_l) V \widetilde{T}_l \right)^* \left( \sum_{l=1}^m M_{T_{\frac{1}{2}}} (a_l) V \widetilde{T}_l \right) \bigg|_E,
\]

where \( M_{T_{\frac{1}{2}}} (a) = \begin{bmatrix} T_{\frac{1}{2}} \widetilde{\Phi}_\rho(a) & \cdots & T_{\frac{1}{2}} \widetilde{\Phi}_\rho(a) \\ 0 & \cdots & 0 \\ \ddots & \ddots & \ddots \\ 0 & 0 & 0 \end{bmatrix} \) and \( V = \begin{bmatrix} \widetilde{V}_{\rho,1} & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & 0 & \widetilde{V}_{\rho,n} \end{bmatrix} \).

From this fact we conclude that \( S(\rho_T) \in CP^\infty(A, M_n(\mathcal{L}_B(E))) \) and by
Proposition 2.1, \( \rho_T \in CP^n_\infty(A, \mathcal{L}_B(E)) \). To show that \( \rho_T \in SCP^n_\infty(A, \mathcal{L}_B(E)) \), let \( \{e_\lambda\}_{\lambda \in \Lambda} \) be an approximate unit for \( A, \xi \in E \) and \( i \in \{1, \ldots, n\} \). Then

\[
\| (\rho_T)_{ii} (e_\lambda) \xi - (\rho_T)_{ii} (e_\mu) \xi \| = \left\| \overline{V}_{\rho,i} \star (T \overline{\Phi}_\rho(e_\lambda) - \overline{\Phi}_\rho(e_\mu)) \overline{V}_{\rho,i} \xi \right\| \\
\leq \left\| \overline{V}_{\rho,i} \star (\| (\Phi_\rho(e_\lambda) - \Phi_\rho(e_\mu)) \|) \right\| \| \overline{V}_{\rho,i} \xi \|,
\]

and since \( \{\Phi_\rho(e_\lambda)V_{\rho,i} \xi\}_{\lambda \in \Lambda} \) is a Cauchy net in \( E \), the net \( \{(\rho_T)_{ii} (e_\lambda)\}_{\lambda \in \Lambda} \) is strictly Cauchy. Therefore \( \rho_T \in SCP^n_\infty(A, \mathcal{L}_B(E)) \).

**Remark 2.4**

1. If \( I_{\tilde{E}_\rho} \) is the identity map on \( \tilde{E}_\rho \), then \( \rho_{I_{\tilde{E}_\rho}} = \rho \).
2. If \( T_1 \) and \( T_2 \) are two positive elements in \( C(\rho) \), then \( \rho_{T_1 + T_2} = \rho_{T_1} + \rho_{T_2} \).
3. If \( T \) is a positive element in \( C(\rho) \) and \( \alpha \) is a positive number, then \( \rho_{\alpha T} = \alpha \rho_T \).

**Remark 2.5**

Let \( T_1 \) and \( T_2 \) be two positive elements in \( C(\rho) \). If \( T_1 \leq T_2 \), then, since

\[
(\rho_{T_2} - \rho_{T_1}) \left[ a_{ij} \right]_{i,j=1}^n = \left[ \overline{V}_{\rho,i} \star (T_2 - T_1) \overline{\Phi}_\rho(a_{ij}) \overline{V}_{\rho,j} \right]_{i,j=1}^n = \rho_{T_2 - T_1} \left[ a_{ij} \right]_{i,j=1}^n
\]

for all \( [a_{ij}]_{i,j=1}^n \in M_n(A) \), \( \rho_{T_1} \leq \rho_{T_2} \).

Let \( \rho \in SCP^n_\infty(A, \mathcal{L}_B(E)) \). We denote by \([0, \rho]\) the set of all strict completely \( n \)-positive linear maps \( \theta \) from \( A \) to \( \mathcal{L}_B(E) \) such that \( \theta \leq \rho \) (that is, \( \rho - \theta \in SCP^n_\infty(A, \mathcal{L}_B(E)) \)) and by \([0, I]_{\rho} \) the set of all elements \( T \) in \( C(\rho) \) such that \( 0 \leq T \leq I_{\tilde{E}_\rho} \).

**Theorem 2.6** The map \( T \rightarrow \rho_T \) from \([0, I]_{\rho}\) to \([0, \rho]\) is an affine order isomorphism.

**Proof.** By Lemma 2.3 and Remarks 2.4 and 2.5, the map \( T \rightarrow \rho_T \) from \([0, I]_{\rho}\) to \([0, \rho]\) is well-defined and moreover, it is affine. To show that this map is injective, let \( T \in [0, I]_{\rho} \) such that \( \rho_T = 0 \). Then \( \overline{V}_{\rho,i} \star T \overline{\Phi}_\rho(a) \overline{\Phi}_\rho(b) \| E \| = 0 \) for all \( a \in A \) and for all \( i, j \in \{1, 2, \ldots, n\} \), and so

\[
\langle T \overline{\Phi}_\rho(a) V_{\rho,j} \xi, \Phi_\rho(b) V_{\rho,j} \eta \rangle = 0
\]
for all $a, b \in A$ for all $\xi, \eta \in E$ and for all $i, j \in \{1, \ldots, n\}$. Taking into account that $\{\Phi_\rho(a)V_{\rho,i}\xi; a \in A, \xi \in E, 1 \leq i \leq n\}$ spans a dense submodule of $E_\rho$, from these facts, Remarks 2.2 and 1.5 and we conclude that $T = 0$.

Let $\theta \in [0, \rho]$. In the same way as in the proof of Lemma 3.4 in [10], we show that there is a bounded linear map $W$ from $E_\rho$ to $E_\theta$ such that

$$W(\Phi_\rho(a)V_{\rho,i}\xi) = \Phi_\theta(a)V_{\theta,i}\xi.$$  

It is not difficult to check that $W$ is a bounded module homomorphism such that $\|W\| \leq 1$, $WV_{\rho,i} = V_{\theta,i}$ for all $i \in \{1, \ldots, n\}$ and $W\Phi_\rho(a) = \Phi_\theta(a)W$ for all $a \in A$. If $\tilde{W}$ is the unique extension of $W$ to a bounded module morphism from $E_\rho$ to $E_\theta$ with $\|\tilde{W}\| = \|W\|$, then clearly $0 \leq \tilde{W}^*\tilde{W} \leq I_{E_\rho}$. Moreover, it is easy to check that $\tilde{W}^*\tilde{W}\Phi_\rho(a) = \tilde{\Phi}_\rho(a)\tilde{W}^*\tilde{W}$ for all $a \in A$, and since

$$\tilde{V}_{\rho,i}^*\tilde{W}^*\tilde{\Phi}_\rho(a)\tilde{V}_{\rho,j}^* = \tilde{V}_{\theta,i}^*\tilde{W}^*\tilde{\Phi}_\theta(a)\tilde{V}_{\theta,j}^*$$

for all $a \in A$ and for all $i, j \in \{1, \ldots, n\}$, $\tilde{W}^*\tilde{W} \in [0, I]_\rho$. Let $T = \tilde{W}^*\tilde{W}$. Then clearly, $\theta = \rho_T$ and thus the map $T \to \rho_T$ from $[0, I]_\rho$ to $[0, \rho]$ is surjective. Therefore the map $T \to \rho_T$ is an affine isomorphism from $[0, I]_\rho$ onto $[0, \rho]$ which preserve the order relation.

**3 Applications of the Radon-Nikodym theorem**

Let $A$ be a locally $C^*$-algebra, let $B$ be a $C^*$-algebra and let $E$ be a Hilbert $C^*$-module over $B$. A strict completely $n$-positive linear map $\rho$ from $A$ to $\mathcal{L}_B(E)$ is said to be pure if for every strict completely $n$-positive linear map $\theta$ from $A$ to $\mathcal{L}_B(E)$ with $\theta \leq \rho$, there is a positive number $\alpha$ such that $\theta = \alpha\rho$.

**Proposition 3.1** Let $\rho \in SCP^\infty_n(A, \mathcal{L}_B(E))$. Then $\rho$ is pure if and only if $[0, I]_\rho = \{\alpha I_{E_{\rho}}; 0 \leq \alpha \leq 1\}$.

**Proof.** First we suppose that $\rho$ is pure. Let $T \in [0, I]_\rho$. By Theorem 2.6, $\rho_T \in [0, \rho]$, and since $\rho$ is pure, $\rho_T = \alpha\rho$ for some positive number. From this fact, Remark 2.4 and Theorem 2.6 we deduce that $T = \alpha I_{E_{\rho}}$ for some $0 \leq \alpha \leq 1$.

Conversely, suppose that $[0, I]_\rho = \{\alpha I_{E_{\rho}}; 0 \leq \alpha \leq 1\}$. Let $\theta \in SCP^\infty_n(A, \mathcal{L}_B(E))$ such that $\theta \leq \rho$. By Theorem 2.6, $\theta = \rho_T$ for some $T \in [0, I]_\rho$, and since $T = \alpha I_{E_{\rho}}$ for some positive number $\alpha$, $\theta = \alpha\rho$ and the proposition is proved.

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Corollary 3.2. A strict completely n-positive linear map \( \rho \) from \( A \) to \( \mathcal{L}_B(E) \) is pure if and only if \( C(\rho) \) consisting of the scalar multipliers of \( I_{\tilde{E}_\rho} \).

We say that two strict completely \( n \)-positive linear maps \( \rho \) and \( \theta \) from \( A \) to \( \mathcal{L}_B(E) \) are unitarily equivalent if the representations of \( A \) induced by \( \rho \) respectively \( \theta \) are unitarily equivalent.

The following proposition is a generalization of Proposition 4.3 in [10].

Proposition 3.3. Let \( A \) be a unital locally \( C^* \)-algebra, let \( B \) be a \( C^* \)-algebra, let \( E \) be a Hilbert \( B \)-module and let \( \rho \in \text{CP}^{\infty}_n(A, \mathcal{L}_B(E)) \). If \( \rho_{ii}, i \in \{1, \ldots, n\} \) are unitarily equivalent pure unital completely positive linear maps from \( A \) to \( \mathcal{L}_B(E) \) and for all \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \) there is a unitary element \( u_{ij} \) in \( A \) such that \( \rho_{ij}(u_{ij}) = I_E \), then \( \rho \) is pure.

Proof. Let \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \). From

\[
\| \Phi_\rho(u_{ij})V_{\rho,j} - V_{\rho,i} \|^2 = \| V_{\rho,j}^*V_{\rho,j} - \rho_{ij}(u_{ij}) - (\rho_{ij}(u_{ij}))^* + V_{\rho,i}^*V_{\rho,i} \| = 0
\]

we deduce that \( \Phi_\rho(u_{ij})V_{\rho,j} = V_{\rho,i} \). Therefore the sets \( \{ \Phi_\rho(a)V_{\rho,i} \xi; a \in A, \xi \in E \} \) and \( \{ \Phi_\rho(a)V_{\rho,j} \xi; a \in A, \xi \in E \} \) generate the same Hilbert submodule of \( E_\rho \), and since \( E_\rho \) is generated by \( \{ \Phi_\rho(a)V_{\rho,i} \xi; a \in A, \xi \in E, 1 \leq i \leq n \} \), this coincides with \( E_\rho \).

Let \( i \in \{1, \ldots, n\} \) and let \( (\Phi_{\rho_{ii}}, E_{\rho_{ii}}, V_{\rho_{ii}}) \) be the KSGNS construction associated with \( \rho_{ii} \). We will show that the representations \( \Phi_\rho \) and \( \Phi_{\rho_{ii}} \) of \( A \) are unitarily equivalent. Since \( \{ \Phi_\rho(a)V_{\rho,i} \xi; a \in A, \xi \in E \} \) spans a dense submodule of \( E_\rho \), \( \{ \Phi_{\rho_{ii}}(a)V_{\rho,i} \xi; a \in A, \xi \in E \} \) spans a dense submodule of \( E_{\rho_{ii}} \) and

\[
\langle \Phi_\rho(a)V_{\rho,i} \xi, \Phi_\rho(b)V_{\rho,i} \eta \rangle = \langle \rho_{ii}(b^*a) \xi, \eta \rangle = \langle V_{\rho_{ii}}^*\Phi_{\rho_{ii}}(b^*a)V_{\rho_{ii}} \xi, \eta \rangle = \langle \Phi_{\rho_{ii}}(a)V_{\rho_{ii}} \xi, \Phi_{\rho_{ii}}(b)V_{\rho_{ii}} \eta \rangle
\]

for all \( a, b \in A \) and for all \( \xi, \eta \in E \), there is a unitary operator \( U_i \) from \( E_{\rho_{ii}} \) to \( E_\rho \) such that \( U_i(\Phi_{\rho_{ii}}(a)V_{\rho_{ii}} \xi) = \Phi_\rho(a)V_{\rho,i} \xi \) [14, Theorem 3.5]. Moreover, \( U_i\Phi_{\rho_{ii}}(a) = \Phi_\rho(a)U_i \) for all \( a \in A \). Then \( \tilde{U}_i \), the unique extension of \( U_i \) to a bounded module homomorphism from \( E_{\rho_{ii}} \) to \( \tilde{E}_\rho \) with \( \| U_i \| = \| \tilde{U}_i \| \), is a unitary element in \( \mathcal{L}_{B^*}(E_{\rho_{ii}}, \tilde{E}_\rho) \).
Let $T \in [0, I]_{\rho_i}$. Then $\tilde{U}_i^* T \tilde{U}_i \in [0, I]_{\rho_i}$, and since $\rho_{ii}$ is pure, by Proposition 3.1, $\tilde{U}_i^* T \tilde{U}_i = \alpha I_{\tilde{E}_{\rho_i}}$ for some positive number $\alpha$. Consequently, $T = \alpha I_{\tilde{E}_\rho}$ and so $\rho$ is pure.

In the following corollary we determine a class of extreme points in the set of all identity preserving completely positive linear maps from a unital locally $C^*$-algebra $A$ to the $C^*$-algebra $M_n(B)$ of all $n \times n$ matrices with elements in the unital $C^*$-algebra $B$. This is a generalization of Corollaries 2.7 in [11] and 4.5 in [10].

**Corollary 3.4** Let $A$ be a unital locally $C^*$-algebra, let $B$ be a unital $C^*$-algebra and let $\rho = [\rho_{ij}]_{i,j=1}^n \in CP^\infty_n(A, B)$. If $\rho_{ii}, i \in \{1, \ldots, n\}$ are unitarily equivalent pure unital completely positive linear maps from $A$ to $B$ and for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$, $\rho_{ij}(1) = 0$ and there is a unitary element $u_{ij}$ in $A$ such that $\rho_{ij}(u_{ij}) = 1$, then the map $\varphi$ from $A$ to $M_n(B)$ defined by $\varphi(a) = [\rho_{ij}(a)]_{i,j=1}^n$ is an extreme point in the set of all identity preserving completely positive linear maps from $A$ to $M_n(B)$.

**Proof.** Let $\varphi_1$ and $\varphi_2$ be two identity preserving completely positive linear maps from $A$ to $M_n(B)$ and let $\alpha \in (0, 1)$ such that $\alpha \varphi_1 + (1 - \alpha) \varphi_2 = \varphi$. Then $\alpha S^{-1}(\varphi_1) + (1 - \alpha) S^{-1}(\varphi_2) = \rho$. From this relation and Propositions 3.3 and 3.1, we conclude that $\alpha S^{-1}(\varphi_1) = \beta_1 \rho$ for some positive number $\beta_1$ and $(1 - \alpha) S^{-1}(\varphi_2) = \beta_2 \rho$ for some positive number $\beta_2$. Consequently, $\alpha \varphi_1 = \beta_1 \varphi$ and $\varphi_2 = \beta_2 \varphi$. From these facts, since $\varphi_1(1) = \varphi_2(1) = \varphi(1) = I_n$, where $I_n$ is the unity matrix in $M_n(B)$, we deduce that $\alpha = \beta_1$ and $1 - \alpha = \beta_2$. Therefore $\varphi_1 = \varphi_2 = \varphi$, and so $\varphi$ is an extreme point in the set of all identity preserving completely positive linear maps from $A$ to $M_n(B)$.

Let $A$ be a unital locally $C^*$-algebra, let $B$ be a $C^*$-algebra and let $E$ be a Hilbert $B$-module. We denote by $CP^\infty_n(A, \mathcal{L}_B(E), I)$ the set of all completely $n$-positive linear maps $\rho = [\rho_{ij}]_{i,j=1}^n$ from $A$ to $\mathcal{L}_B(E)$ such that $\rho_{ii}(1) = I_E$ for all $i \in \{1, \ldots, n\}$ and $\rho_{ij}(1) = 0$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

The following theorem is a generalization of Theorems 3.8 in [18] and 4.6 in [10].

**Theorem 3.5** Let $\rho \in CP^\infty_n(A, \mathcal{L}_B(E), I)$. Then $\rho$ is an extreme point in the set $CP^\infty_n(A, \mathcal{L}_B(E), I)$ if and only if the map $T \rightarrow [\tilde{V}_{\rho,i}^* T \tilde{V}_{\rho,j}]_{i,j=1}^n$ from $C(\rho)$ to $M_n(\mathcal{L}_{B^*}(\tilde{E}))$ is injective.
Proof. Suppose that $\rho$ is an extreme point in the set $CP^n_{\alpha}(A, \mathcal{L}_B(E), I)$ and $T$ is an element in $C(\rho)$ such that $V_{\rho,i}^*TV_{\rho,i} = 0$ for all $i, j \in \{1, ..., n\}$. Since $V_{\rho,j}^*T^*V_{\rho,i} = \left(V_{\rho,i}^*TV_{\rho,j}\right)^*$ for all $i, j \in \{1, ..., n\}$, we can suppose that $T = T^*$. It is not difficult to check that there are two positive numbers $\alpha$ and $\beta$ such that $\frac{1}{4}I_{E_{\rho}} \leq \alpha T + \beta I_{E_{\rho}} \leq \frac{3}{4}I_{E_{\rho}}$. Moreover, $\beta \in (0, 1)$. Let $T_1 = \frac{\sqrt{3}}{2}T + I_{E_{\rho}}$ and $T_2 = I_{E_{\rho}} - \frac{\alpha}{4}T$. Clearly, $T_k, k \in \{1, 2\}$ are two positive elements in $C(\rho)$. Then $\rho T_k \in CP^n_{\alpha}(A, \mathcal{L}_B(E)), k \in \{1, 2\}$ and since

$$(\rho T_k)_{ij}(1) = \left|V_{\rho,i}^*)T_kV_{\rho,j}\right|_E = \left|V_{\rho,i}^*V_{\rho,j}\right|_E = V_{\rho,i}^*V_{\rho,j} = \begin{cases} I_E & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for all $i, j \in \{1, ..., n\}$ with $i \neq j$, and $k \in \{1, 2\}, \rho T_k \in CP^n_{\alpha}(A, \mathcal{L}_B(E), I)$ for each $k \in \{1, 2\}$. A simple calculus shows that $\beta \rho T_1 + (1 - \beta) \rho T_2 = \rho$, and since $\rho$ is an extreme point, $\rho T_1 = \rho T_2 = \rho$. But $\rho T_1 = \frac{\sqrt{3}}{2} \rho T + \rho$ and $\rho T_2 = \rho - \frac{\alpha}{4} \rho T$. Therefore $\rho T = 0$ and by Theorem 2.6, $T = 0$.

Conversely, suppose that the map $T \rightarrow \left[V_{\rho,i}^*TV_{\rho,j}\right]_{i,j=1}^n$ from $C(\rho)$ to $M_n(\mathcal{L}_{B^{\alpha}}(E))$ is injective. Let $\theta, \sigma \in CP^n_{\alpha}(A, \mathcal{L}_B(E), I)$ and $\alpha \in (0, 1)$ such that $\alpha \theta + (1 - \alpha) \sigma = \rho$. By Theorem 2.6, there are two elements $T_1$ and $T_2$ in $[0, I]_{\rho} \subseteq C(\rho)$, such that $\alpha \theta = \rho T_1$ and $(1 - \alpha) \sigma = \rho T_2$. Then

$$\left|V_{\rho,i}^*T_1V_{\rho,j}\right|_E = (\rho T_1)_{ij}(1) = \alpha \theta_{ij}(1) = \begin{cases} \alpha I_E & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and

$$\left|V_{\rho,i}^*T_2V_{\rho,j}\right|_E = (\rho T_2)_{ij}(1) = (1 - \alpha) \sigma_{ij}(1) = \begin{cases} (1 - \alpha) I_E & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$
References


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