is even and periodic in $x$ with period 1. Therefore it is the sum of a cosine series

$$F(x, y) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos(2n\pi x),$$

with $A_n = 2 \int_{-\infty}^{\infty} e^{-x^2} \cos(2\pi nx) \, dx$. It follows that

$$F(x, y) = \sqrt{\frac{\pi}{y}} \left(1 + 2 \sum_{n=1}^{\infty} e^{-\pi^2 n^2 / y} \cos(2\pi nx)\right). \tag{2}$$

Substituting $x = 1/2$ and $y = \pi$ into the expression for $F(x, y)$ given in (1) leads to

$$F(1/2, \pi) = \sum_{k=-\infty}^{\infty} e^{-(k+1/2)^2\pi} = \sum_{k=-\infty}^{-1} e^{-(k+1/2)^2\pi} + e^{-\pi/4} + \sum_{k=1}^{\infty} e^{-(k+1/2)^2\pi}$$

$$= \sum_{k=2}^{\infty} e^{-(k-1/2)^2\pi} + 2e^{-\pi/4} + \sum_{k=1}^{\infty} e^{-(k+1/2)^2\pi} = 2 \sum_{k=0}^{\infty} e^{-(k+1/2)^2\pi}.$$

Substituting into (2), we obtain

$$F(1/2, \pi) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-\pi n^2} = \sum_{n=0}^{\infty} (-1)^n e^{-\pi n^2} + \sum_{n=1}^{\infty} (-1)^n e^{-\pi n^2}$$

$$= \sum_{n=0}^{\infty} (-1)^n e^{-\pi n^2} + \sum_{n=0}^{\infty} (-1)^{n+1} e^{-\pi (n+1)^2}.$$

Equating these two expressions for $F$, we find that

$$2 \sum_{k=0}^{\infty} e^{-(k+1/2)^2\pi} = \sum_{n=0}^{\infty} (-1)^n e^{-\pi n^2} + \sum_{n=0}^{\infty} (-1)^{n+1} e^{-\pi (n+1)^2}.$$

Multiplication of both sides by $e^{\pi/2}$ leads to

$$2 \sum_{k=0}^{\infty} e^{\pi/4} e^{-k(k+1)\pi} = \sum_{n=0}^{\infty} (-1)^n e^{-\pi n(n+1)} \left(e^{\pi(n+1)/2} - e^{-n\pi - \pi/2}\right).$$

Applying the definition of the hyperbolic sine, we obtain the desired equation.

Also solved by R. Chapman (U.K.), J. Grivaux (France), O. Koubi (Syria), G. Lamb, M. A. Prasad (India), O. G. Ruehr, A. Stadler (Switzerland), R. Stong, J. Sun, FAU Problem Solving Group, GCHQ Problem Solving Group, and the proposer.

**A Variant Intermediate Value**

**11290** [2007, 359]. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal Highschool, Constanza, Romania.* Let $f$ and $g$ be continuous real-valued functions on $[0, 1]$. Prove that there exists $c$ in $(0, 1)$ such that

$$\int_{x=0}^{1} f(x) \, dx \int_{x=0}^{c} x g(x) \, dx = \int_{x=0}^{1} g(x) \, dx \int_{x=0}^{c} x f(x) \, dx.$$
Solution by Kenneth F. Andersen, University of Alberta, Edmonton, AB, Canada. Observe first that if \( h(x) \) is continuous on \([0, 1] \) and \( H(x) = \int_0^x yh(y) \, dy \), then \( H(x) \) is continuous on \([0, 1] \) with \( \lim_{x \to 0^+} H(x)/x = 0 \), so an integration by parts yields

\[
\int_0^1 h(x) \, dx = \int_0^1 \frac{xh(x)}{x} \, dx = \frac{H(x)}{x} \bigg|_0^1 + \int_0^1 \frac{H(x) \, dx}{x^2} = H(1) + \int_0^1 \frac{H(x) \, dx}{x^2} = \lim_{x \to 1^-} H(x) + \int_0^1 \frac{H(x) \, dx}{x^2}. \tag{1}
\]

Now suppose in addition that \( \int_0^1 h(x) \, dx = 0 \). By (1), \( H(x) \) cannot be positive for all \( x \) in \((0, 1)\), nor can it be negative for all \( x \) in \((0, 1)\). Thus by the Intermediate Value Theorem there is a \( c_h \in (0, 1) \) such that \( H(c_h) = 0 \). Now the required result may be deduced: if \( \int_0^1 f(x) \, dx = 0 \), then the result holds with \( c = c_f \); if \( \int_0^1 g(x) \, dx = 0 \), then the result holds with \( c = c_g \). Otherwise the result holds with \( c = c_h \), where

\[
h(x) = \frac{f(x)}{\int_0^1 f(y) \, dy} - \frac{g(x)}{\int_0^1 g(y) \, dy}.
\]

Editorial comment. (i) The functions \( f \) and \( g \) need not be continuous—it is sufficient that they be integrable. This was observed by Botsko, Pinelis, Schilling, and Schmuland. (ii) Keselman, Martin, and Pinelis noted that \( \int_0^1 xf(x) \, dx \) and \( \int_0^1 xg(x) \, dx \) can be replaced with \( \int_0^1 \phi(x)f(x) \, dx \) and \( \int_0^1 \phi(x)g(x) \, dx \), where \( \phi(x) \) satisfies suitable conditions—roughly speaking, that \( \phi \) is differentiable and strictly monotonic, although the specific conditions vary from one of these solvers to another.


Double Integral

11295 [2007, 452]. Proposed by Stefano Siboni, University of Trento, Trento, Italy. For positive real numbers \( \epsilon \) and \( \omega \), let \( M \) be the mapping of \([0, 1) \times [0, 1) \) into itself defined by \( M(x, y) = (\{2x\}, \{y + \omega + \epsilon x\}) \), where \( \{u\} \) denotes \( u - \lfloor u \rfloor \), the fractional part of \( u \). For integers \( a \) and \( b \), let \( e_{a,b}(x, y) = e^{2\pi i(ax+by)} \). Let

\[
C_n(a, b; p, q) = \int_{x=0}^1 \int_{y=0}^1 e_{a,b}(M^n(x, y)) e_{p,q}(x, y) \, dx \, dy.
\]

Show that \( C_n(a, b; p, q) = 0 \) if \( q \neq b \), while \( C_n(a, b; p, b) \) is given by

\[
(-1)^a e_{b,b}(\omega n, \epsilon n/2) \frac{\sin\left[\frac{\pi(a+\epsilon b - 2^{-n}(p + \epsilon b))}{\pi(a + \epsilon b - 2^{-n}(p + \epsilon b))}\right]}{\prod_{j=0}^{n-1} \cos\left[\frac{\pi \epsilon b - 2^{-j}(p + \epsilon b)}{2}ight]}.
\]

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. Let \( e(z) = e^{2\pi i z} \). For \( n \geq 1 \) we have

\[
M^n(x, y) = \left(\{2^n x\}, \{y + n\omega + \epsilon (\{x\} + \{2x\} + \cdots + \{2^{n-1} x\})\}\right).
\]

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