SOLUTION TO AMM PROBLEM # 11369

ÁNGEL PLAZA AND JOSÉ MIGUEL PACHECO

Proposed by DONALD KNUTH, Stanford University, Stanford, CA.

Problem . # 11369 Show that for all real \( t \), and all \( \alpha \geq 2 \),

\[ e^{\alpha t} + e^{-\alpha t} - 2 \leq (e^t + e^{-t})^\alpha - 2^\alpha. \]

Solution: It is clear that the equality holds for \( t = 0 \) and any \( \alpha \geq 2 \), and also for any real \( t \) and \( \alpha = 2 \). Let us suppose then that \( t \neq 0 \) and \( \alpha > 2 \). Since \( x = e^t > 0 \) in this case, the inequality may be written as

\[ x^\alpha + x^{-\alpha} - 2 < \left( x + \frac{1}{x} \right)^\alpha - 2^\alpha. \]  

(1)

Also, since \( x \cdot x^{-1} = 1 \) it can be supposed that \( x > 1 \).

Note that if \( g(x) = x^\alpha + x^{-\alpha} \) and \( f(x) = \left( x + \frac{1}{x} \right)^\alpha \), then Eq. (1) may be written as

\[ g(x) - g(1) < f(x) - f(1), \]

(2)

or, equivalently,

\[ \frac{g(x) - g(1)}{f(x) - f(1)} < 1. \]  

(3)

Now, by the Lagrange Theorem, the Left-Hand Side of Eq. (3) is \( \frac{g'(c)}{f'(c)} \), for some real \( c \) such that \( 1 < c < x \).

Note that \( \frac{g'(c)}{f'(c)} < 1 \) \( \Leftrightarrow \) \( g'(c) < f'(c) \). That is, using \( x \) instead of \( c \),

\[ \alpha x^{\alpha-1} - \alpha \frac{1}{x^{\alpha+1}} < \alpha \left( x + \frac{1}{x} \right)^{\alpha-1} \left( 1 - \frac{1}{x^2} \right) \]  

(4)

\[ x^{\alpha-1} \left[ 1 - \frac{1}{x^{2\alpha}} \right] < x^{\alpha-1} \left( 1 + \frac{1}{x^2} \right)^{\alpha-1} \left( 1 - \frac{1}{x^2} \right) \]  

(5)

\[ \frac{1}{x^2} = y \] gives \( 0 < y < 1 \) and Eq. (5) reads:

\[ 1 - y^{\alpha} < (1 + y)^{\alpha-1}(1 - y) = (1 + y)^{\alpha-1} - y(1 + y)^{\alpha-1} \]  

(6)

\[ 1 - (1 + y)^{\alpha-1} < y^\alpha - y(1 + y)^{\alpha-1} = y \left[ y^{\alpha-1} - (1 + y)^{\alpha-1} \right] \]  

(7)

Date: October 31, 2008.
Let us consider function $F(y) = y^{\alpha - 1}$. $F$ is strictly convex, since $F''(y) = (\alpha - 1)(\alpha - 2)y^{\alpha - 3} > 0$, for $y > 0$ and $\alpha > 2$. If denote by $\Delta_F(x, y) = \frac{F(y) - F(x)}{y - x}$ the divided difference of function $F$, then Eq. (8) may be understood as:

\[(9) \quad \frac{\Delta_F(1, 1 + y)}{\Delta_F(y, 1 + y)} > y\]

which is equivalent to

\[(10) \quad \frac{\Delta_F(1, 1 + y)}{\Delta_F(y, 1 + y)} > 1 \iff \Delta_F(1, 1 + y) > \Delta_F(y, 1 + y)\]

Now, we use the following lemma [1]:

**Lemma.** A function $F : (a, b) \rightarrow \mathbb{R}$ is convex (strictly convex) if and only if its divided difference $\Delta_F(x, y)$ is increasing (strictly increasing) in both variables.

Inequality (10) may be illustrated by the following figure, considering that $\Delta_F(1, 1 + y)$ is the slope of the line passing through points $B$ and $C$, while $\Delta_F(y, 1 + y)$ is the slope of the line passing through points $A$ and $C$:

\[y = x^{\alpha - 1}\]

\[y = x\]

Note, also, that for the case $\alpha = 2$, function $y = x^{\alpha - 1}$ into the previous figure is precisely $y = x$ and in this case we have the equality. □

**References**