In this note, we present an affirmative answer to a question presented in the paper “Some inequalities in inner product spaces related to the generalized triangle inequality” by S.S. Dragomir et al. [Appl. Math. Comput. 217 (18) (2011) 7462–7468].

In a recent paper [1], Dragomir et al. proposed the open problem of whether constant $\frac{1}{2}$, appearing in the following two theorems, is the best possible. In this short note, we answer in affirmative sense this question.

**Theorem 1** [1, Theorem 6]. Let $(H; \langle , \rangle)$ be an inner product space, $x_i \in H$, for all $i \in \{1, \ldots, n\}$ and $p_i \geq 0$ with $\sum_{i=1}^{n} p_i = 1$. Suppose there exist constants $r_i > 0$, $i \in \{1, \ldots, n\}$, so that

$$\left\| x_i - \sum_{j=1}^{n} p_j x_j \right\| \leq r_i$$

for all $i \in \{1, \ldots, n\}$. Then

$$0 \leq \sum_{i=1}^{n} p_i \left\| x_i \right\| - \left\| \sum_{i=1}^{n} p_i x_i \right\| \leq \frac{1}{2} \frac{\sum_{i=1}^{n} p_i r_i^2}{\left\| \sum_{i=1}^{n} p_i x_i \right\|}$$

provided that $\sum_{i=1}^{n} p_i x_i \neq 0$. □

**Theorem 2** [1, Theorem 7]. Let $x_i$, $p_i$ and $r_i$ be as in the statement of previous theorem. Then

$$0 \leq \left( \sum_{i=1}^{n} p_i \left\| x_i \right\|^2 \right)^{\frac{1}{2}} - \left\| \sum_{i=1}^{n} p_i x_i \right\| \leq \frac{1}{2} \frac{\sum_{i=1}^{n} p_i r_i^2}{\left\| \sum_{i=1}^{n} p_i x_i \right\|}.$$  □

Our example is the following.

Let us consider the Euclidean space $\mathbb{R}^2$, $x_1 = (1, 0)$ and $x_2 = (x, \beta)$ with $\left\| x_i \right\|^2 = 1$ for $i = 1, 2$, that is $x^2 + \beta^2 = 1$. We choose $p_1 = \frac{1}{2}$ and $p_2 = \frac{1}{2}$. Then $p_1 \left\| x_1 \right\| + p_2 \left\| x_2 \right\| = 1$ and
\[ p_1 x_1 + p_2 x_2 = \left( \frac{1}{4} + \frac{3}{4} \alpha \right), \]

\[ \|x_1 - (p_1 x_1 + p_2 x_2)\|^2 = \left( \frac{3}{4} - \frac{3}{4} \alpha \right)^2 + \left( \frac{3}{4} \beta \right)^2 = \frac{18}{16} (1 - \alpha) = r_1^2, \]

\[ \|x_2 - (p_1 x_1 + p_2 x_2)\|^2 = \left( -\frac{1}{4} + \frac{1}{4} \alpha \right)^2 + \left( \frac{1}{4} \beta \right)^2 = \frac{2}{16} (1 - \alpha) = r_2^2, \]

\[ \|p_1 x_1 + p_2 x_2\|^2 = \left( \frac{1}{4} + \frac{3}{4} \alpha \right)^2 + \left( \frac{3}{4} \beta \right)^2 = \frac{10}{16} + \frac{6}{16} \alpha. \]

In this case, inequalities (1) and (2) are:

\[ 1 - \frac{1}{4} \sqrt{10 + 6 \alpha} \leq \frac{3}{2} \frac{1 - \alpha}{\sqrt{10 + 6 \alpha}} \]

or, equivalently,

\[ \frac{1 - \frac{1}{4} \sqrt{10 + 6 \alpha}}{\frac{3}{2} (1 - \alpha)} \leq \frac{1}{2}. \quad (3) \]

In order to prove that the previous inequality is sharp, assume that there exists a constant \( c > 0 \) such that inequality (3) holds with \( c \), i.e.

\[ \frac{1 - \frac{1}{4} \sqrt{10 + 6 \alpha}}{\frac{3}{2} (1 - \alpha)} \leq c. \quad (4) \]

Now, since \( \alpha \in (0, 1) \) we may consider \( \alpha = 1 - \varepsilon \). Letting \( \varepsilon \to 0 \) and taking into account L'Hopital rule we get:

\[ \lim_{\varepsilon \to 0} \frac{1 - \frac{1}{4} \sqrt{16 - 6 \varepsilon}}{\frac{3 \varepsilon}{2 \sqrt{16 - 6 \varepsilon}}} = \lim_{\varepsilon \to 0} \frac{8}{3 \varepsilon} \left( 1 - \frac{1}{4} \sqrt{16 - 6 \varepsilon} \right) = \frac{1}{2}. \]

Therefore, constant \( \frac{1}{2} \) is the best possible in Theorems 1 and 2. \( \square \)

References