Proof Without Words: Sum of Triangular Numbers

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Theorem.

\[ T_n = 1 + 2 + \cdots + n = \frac{n(n + 1)}{2} = \left( \begin{array}{c} n+1 \\ 2 \end{array} \right) \Rightarrow \sum_{k=1}^{n} T_k = \frac{n(n + 1)(n + 2)}{6}. \]

Proof.

\[ \sum_{k=1}^{n} T_k = \begin{array}{cccccc}
1 & 2 & 3 & \cdots & n \\
T_n & T_3 & T_2 & T_1 & & \\
& 1 & 2 & 3 & & \\
& 1 & 2 & & & \\
& 1 & & & & \\
\end{array} \]

\[ 3 \sum_{k=1}^{n} T_k = \begin{array}{cccccc}
1 \times n & \cdots & n \times 1 \\
2 \times (n-1) & \cdots & (n-1) \times 1 \\
1 \times (n-1) & \cdots & (n-1) \times (n-1) \\
\end{array} \]

\[ \frac{n(n+1)}{2} \]
This proof is close to, and it can be seen as a variation of, Zerger’s proof [1], which also appears on page 94 of Nelsen’s compendium of PWWs [2].

REFERENCES

Summary. The triangular numbers are given by the following explicit formulas: 
\[ T_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} = \binom{n+1}{2} \]. Here it is proved visually that
\[
\sum_{k=1}^{n} T_k = \frac{n(n+1)(n+2)}{6}.
\]

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From the Files of Past MAGAZINE Editors

As chairman of the MAA’s Publications Committee, Ed Beckenbach asked then MAGAZINE co-editors Lynn Arthur Steen (LAS) and J. Arthur Seebach, Jr. (AS) what ideas they had for improving the MAGAZINE. They wanted to make it more public-oriented, but they realized they had no writers and no audience for such a magazine. Instead, they made some cosmetic changes, like putting something other than the table of contents on the cover, itself a controversial decision.

LAS and AS wanted to have articles start at the top of a page instead of simply starting where the previous article ended; ideally they would start at the top of a right-hand page. They needed some short things (called filler) to insert to fill space at the ends of some articles. At a suggestion of Roger Nelsen, they began including some Proofs without Words as filler. Proofs without Words have been a mainstay of the MAGAZINE ever since.